finite collection of sets be $\{H_{\alpha_1}, H_{\alpha_2}, \ldots, H_{\alpha_N}\}$ such that $Y \subset \bigcup_{i=1}^{N} H_{\alpha_i}$

$$\Rightarrow Y = Y \cap \left(\bigcup_{i=1}^{N} H_{\alpha_i}\right) = \bigcup_{i=1}^{N} \left(Y \cap H_{\alpha_i}\right) = \bigcup_{i=1}^{N} G_{\alpha_i}$$

$\Rightarrow \{G_{\alpha_i} : i = 1, 2, \ldots, N\}$ is a finite subcollection of sets in $\{G_\alpha : \alpha \in I\}$ which covers Y.

4.2. Sequential Compactness

4.2.1 Definition: Let (X, d) be a metric space

(a) The space (X, d) is said to be sequentially compact if every sequence in X has a convergent subsequence.

(b) A subspace (Y, d_Y) of (X, d) is said to be sequentially compact if it is sequentially compact as a metric space in its own right

(c) A subset $Y \subset X$ is said to be sequentially compact if it is sequentially compact as a metric subspace

5.2.2 Definition: A metric space (X, d) is said to have Bolzano Weierstrass Property (BWP) if every infinite subset of X has a limit point.

We now state some theorems without proof:

4.2.3 Theorem: A metric space is sequentially compact if and only if it has the BWP

4.2.4 Theorem: A compact metric space has the BWP

4.2.5 Corollary: A compact metric space is sequentially compact

4.2.6 Theorem: A compact subset of a metric space is closed and bounded.

The converse of Theorem 4.2.6 need not be true. Consider for instance an infinite subset of A of a discrete metric space X_d (Example 4.1.4(3)). The set is clearly closed and bounded but not compact. However, the converse is true in special case

4.2.7 Theorem (Heine Borel Theorem in $\mathbb{R}$)

A subset of the usual metric space $\mathbb{R}_u$ is closed and bounded if and only if it is compact.

Proof: Let F be a closed and bounded subset of $\mathbb{R}$. We shall prove that every open cover of F admits a finite subcover.

Case(i) Suppose F is a closed interval, say, $[a,b]$. Also let $\mathcal{C}$ be an open cover of $[a,b]$. Take E to be the set of number $x \leq b$ such that the interval $[a,x]$ is contained in the union of a finite number of sets in $\mathcal{C}$. Clearly, E is non-empty set as $a \in E$ and is bounded above by b. So, it has the least upper bound, say, c. Since $c \in [a,b]$, $\exists$ an open G in $\mathcal{C}$ which contains c. Since G is open $\exists$ an $\varepsilon > 0$ such that $(c-\varepsilon, c+\varepsilon) \subset G$. Now $c-\varepsilon$ is not an upper ~~of~~ bound of E, and hence $x > c-\varepsilon$ for some $x \in E$. Since $x \in E$, $[a,x]$ is contained in the union of a finite number of of sets in $\mathcal{C}$. Consequently, the finite subcollection obtained by adding one more set G to the finite number of sets, already required to cover $[a,x]$, covers

$[a, c+\varepsilon]$. Thus $[c, c+\varepsilon] \subset E$ if each point of $[c, c+\varepsilon]$ is less than or equal to b. Since no point of $[c, c+\varepsilon]$ except c can belong to E, we must have $c=b$ and $b \in E$. Thus $[a,b]$ can be covered by a collection consisting of finite number of sets in $\mathscr{C}$. Thus the result is proved when $F = [a,b]$.

Case (ii) Suppose F is a closed and bounded subset of $\mathbb{R}$. Let $\mathscr{C}$ be an open cover of F. Being a bounded set, we enclose F in a closed interval $[a,b]$. Let $\mathscr{D}$ be the collection obtained by adding F^c to $\mathscr{C}$; i.e., $\mathscr{D} = \mathscr{C} \cup \{F^c\}$. Clearly $\mathscr{D}$ is an open cover of $\mathbb{R}$ as $\mathbb{R} = F \cup F^c \subset (\bigcup_{G\in\mathscr{C}} G) \cup F^c$ and hence of $[a,b]$. By case (i), $\exists$ a finite subcover $\mathscr{D}'$ of $\mathscr{D}$ which covers $[a,b]$ and hence $[a,b]$ and hence. Since $F \cap F^c = \phi$, $\mathscr{D}' - \{F^c\}$ covers F. However $\mathscr{D}' - \{F^c\}$ is a finite subcollection of $\mathscr{C}$.

Hence F is compact.

The converse follows as a particular case of Theorem 4.2.6.

Another theorem we state without proof

4.2.8 Theorem : Every sequentially compact metric space is compact.

So, this proves the equivalence of compactness and

4.3 ~~Compact~~ Compactness and finite intersection property

4.3.1 Definition: Let (X,d) be a metric space and $\mathcal{C}$ be a collection of subsets of X. $\mathcal{C}$ is said to have finite intersection property (f.i.p.) if every finite subcollection has non-empty intersection.

4.3.1 Theorem: Let (X,d) be a metric space. Then X is compact if and only if every collection of closed subsets of X having f.i.p. has non-empty intersection.

Proof: Let X be compact and $\mathcal{C} = \{F_\alpha : \alpha \in I\}$ be any family of closed sets with f.i.p. We shall prove that $\bigcap_{\alpha\in I} F_\alpha \neq \phi$

On the contrary, if $\bigcap_{\alpha\in I} F_\alpha = \phi$, then $\bigcup_{\alpha\in I}(X-F_\alpha) = X - \bigcap_{\alpha\in I} F_\alpha$ (De Morgan's law)

$= X \quad (\because \bigcap_{\alpha\in I} F_\alpha = \phi)$

$\Rightarrow \mathcal{D} = \{X - F_\alpha : \alpha\in I\}$ is an open cover of X

since $X - F_\alpha$ is open for each $\alpha \in I$. But X is compact,

$\mathcal{D}$ has a finite subcover $\mathcal{D}' = \{X - F_{\alpha_i} : i = 1,2,\dots,N\}$

Now $\bigcup_{i=1}^{N}(X - F_{\alpha_i}) = X \Rightarrow X - \bigcap_{i=1}^{N} F_{\alpha_i} = X$

$\Rightarrow \bigcap_{i=1}^{N} F_{\alpha_i} = \phi$. This shows that $\mathcal{C}$ has no f.i.p., which is a contradiction. Hence $\bigcap_{\alpha\in I} F_\alpha \neq \phi$.

Conversely, let X be not compact. Then, $\exists$ an open cover $\mathcal{C} = \{G_\alpha : \alpha\in I\}$ of X which has no finite subcover. In other words, for every finite subcollection $\mathcal{C}' = \{G_{\alpha_i} : i = 1,2,\dots,N\}$ of $\mathcal{C}$, we have

$\bigcup_{i=1}^{N} G_{\alpha_i} \neq X \Rightarrow \bigcap_{i=1}^{N}(X - G_{\alpha_i}) \neq \phi$. Write $F_\alpha = X - G_\alpha$, $\alpha\in I$

Then F_α is closed, for each $\alpha\in I$ and $\{F_\alpha : \alpha\in I\}$ is a family of closed set with f.i.p because $\bigcap_{i=1}^{N} F_{\alpha_i} = \bigcap_{i=1}^{N}(X - G_{\alpha_i}) \neq \phi$

But $\bigcup_{\alpha\in I} G_\alpha = X$ and so $\bigcap_{\alpha\in I} F_\alpha = \bigcap_{\alpha\in I}(X - G_\alpha) = X - \bigcup_{\alpha\in I} G_\alpha = \phi$, a contradiction. Hence X must be compact.