4.4 Continuous functions on compact sets

4.4.1 Theorem: Let (X, d_1) and (Y, d_2) be metric spaces and $f: X \to Y$ be a continuous function. If $A \subset X$ is compact in X, then $f(A)$ is compact in Y.

Proof: Let A be compact. Then by Theorem 4.2.5, A is sequentially compact. Consider an arbitrary sequence $\{y_n\}$ in $f(A)$. For each y_n, we can choose $x_n \in A$ such that $f(x_n) = y_n$. Thus, we get a sequence $\{x_n\}$ in A. But A being sequentially compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ and the continuity of f implies that the corresponding subsequence $\{y_{n_k}\}$ of $\{y_n\}$ is convergent. But $\{y_n\}$ is arbitrary sequence in $f(A)$. Hence $f(A)$ is sequentially compact and again by Theorem 4.2.8, $f(A)$ is compact.

4.4.2 Corollary: Let (X, d_1) and (Y, d_2) be metric spaces and $f: X \to Y$ be a continuous function. If X is compact, then $f(X)$ is bounded.

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5. Connectedness

5.1. Separated sets

5.1.1 Definition: Let (X, d) be a metric space and $A, B \subset X$. The sets A and B are said to be separated if

$$A \cap \bar{B} = \phi \text{ and } \bar{A} \cap B = \phi.$$

5.1.2 Examples

1. In the usual metric space $\mathbb{R}_u$

(a) the sets $A = (0,1)$ and $B = (1,2)$ are separated.

(b) the sets $A = (0,1]$ and $B = (1,2)$ are not separated.

2. In the discrete metric space $\mathbb{R}_d$, the set $A = (0,1]$ and $B = (1,2)$ are separated. In general, any two disjoint sets in $\mathbb{R}_d$ are separated (verify!)

5.1.3 Theorem: Let (X,d) be a metric space and $C, D \subset X$ be separated sets. If $A \subset C$ and $B \subset D$ then A and B are separated.

Proof: Prove it.

5.1.4 Let (X,d) be a metric space and $A, B \subset X$. If $d(A,B) > 0$, then A and B are separated sets

Proof: Let $d(A,B) = \lambda$. Then

$$\lambda = \inf.\{d(x,y) : x \in A, y \in B\} > 0$$

By the definition of infimum, we have

$$d(x,y) \geq \lambda \quad \forall x \in A \text{ and } \forall y \in B$$

If $x \in A$, then x can not be a limit point of B since $B_{\lambda/2}$ contains no point of B. So, $A \cap \overline{B} = \phi$. Similarly, it can be proved that $\overline{A} \cap B = \phi$. Hence A and B are separated sets.

Remark: The converse of 5.1.4 is not true

5.1.5 Example: In the usual metric space $\mathbb{R}_u$, the sets $A = \{x : x < 0\}$ and $B = \{x : x > 0\}$ are separated while $d(A,B) = 0$

5.1.6 Theorem: Let (X,d) be a metric space and $A, B \subset X$

are disjoint sets. Then :

(a) If A, B both are closed, then A and B are separated.

(b) If A, B both are open, then A and B are separated.

Proof : (a) It follows by using the definition of separated sets and the fact that $\bar{A} = A$ and $\bar{B} = B$

(b) If possible, let, A and B are not separated. Then either $A \cap \bar{B} \neq \phi$ or $\bar{A} \cap B \neq \phi$. Suppose $A \cap \bar{B} \neq \phi$. Let $x \in A \cap \bar{B}$. Then $x \in A$ and $x \in \bar{B}$

So, $B_r(x) \subset A$ and $(B_r(x) - \{x\}) \cap B \neq \phi$, for some $r > 0$. Consequently, $A \cap B \neq \phi$, a contradiction. Hence A and B are separated. Similar proof if $\bar{A} \cap B \neq \phi$.

5.1.7 In a metric space any two sets, either both open or both closed are separated if and only if they are disjoint.

5.1.8 Let (X, d) be a metric space, $G \subset X$ and $G = A \cup B$, where A and B are separated sets.

(a) If G is open, then A and B are open

(b) If G is closed, then A and B are closed.

Proof : (a) Assume that A and B are non-empty sets; for if $A = \phi$ then $B = G$ and hence both A and B are open.

Let $x \in A$. Then $x \in G$. Since G is open, $B_r(x) \subset G$, for some $r > 0$. Moreover

$$A \cap \bar{B} = \phi \text{ and } x \in A \Rightarrow x \notin \bar{B}$$

$\Rightarrow \exists$ an open ball $B_{r_1}(x)$ with $r_1 \leq r$ such that $B_{r_1}(x) \cap B = \phi$

$\Rightarrow B_{r_1}(x) \subset A$

$\Rightarrow x \in A^0$. Since $x \in A$ is arbitrary, each point of A is an interior point of A. Hence A is an open set. Similarly, we can prove that B is an open set.

(b) we have

$$\bar{A} = \bar{A} \cap (\bar{A} \cup \bar{B})$$
$$= \bar{A} \cap (A \cup B) \quad (\because A\cup B (= G) \text{ is closed})$$
$$= (\bar{A} \cap A) \cup (\bar{A} \cap B)$$
$$= A \cup \phi = A$$

This proves that A is closed. Similarly, we can prove that B is closed

5.2 Disconnected and Connected sets.

5.2.1 Definition: A metric space (X, d) is said to be disconnected if X can be expressed as the union of two non-empty separated sets. Furthermore, X is said to be connected if it is not disconnected.

We give below some characterisations of a disconnected (hence connected) metric space

5.2.2 Theorem: Let (X, d) be a metric space. The following statements are equivalent:

(a) (X, d) is disconnected

(b) X is expressible as the union of two disjoint, non-empty closed sets.

(c) X is expressible as the union of two disjoint, nonempty, open sets.

(d) $\exists$ a non-empty, proper subset of X which is both open and closed.