Notes on CC13 for SEMESTER VI
(Metric Space & Complex Analysis)

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Books followed : 1. Elements of Metric Spaces - M.N. Mukherjee

2. Metric Spaces - S. Shirali & H.L. Vasudeva

3. Topology of Metric Spaces - S. Kumaresan

4. Metric Spaces - P.K. Jain & K. Ahmed

1. Introductory Concepts

In analysis, we are concerned mainly with two elementary concepts: (i) convergent sequence in $\mathbb{K}$ and (ii) continuous functions with domains and ranges in $\mathbb{K}$ ($\mathbb{K}$ is the field of real numbers $\mathbb{R}$, or the field of complex numbers $\mathbb{C}$). We note that each of these two notions depends precisely on the concept of the absolute value $|x - x_0|$ of the difference between the number x and x_0 in $\mathbb{K}$. Many of the properties of convergent sequences and continuous functions depend only on the properties of this distance and not on the algebraic nature of real (or complex) number systems. We write $d(x,y) = |x-y|$, the distance between x and y. The following properties of the distance are well known:

(i) $|x-y| \geq 0$

(ii) $|x-y| = 0 \Leftrightarrow x = y$

(iii) $|x-y| = |y-x|$

(iv) $|x-y| \leq |x-z| + |z-y|$

We want to generalize the concept of distance by taking any non-empty set (instead of $\mathbb{R}$ or $\mathbb{C}$) in such a way that we

We can generalise the notions of convergence of sequences and continuity of functions (real or complex) to a more general situation.

1.1 Definition and Examples of Metric spaces.

1.1.1 Definition

Let X be a non-empty set. A function $d: X \times X \to \mathbb{R}$ is said to be a metric on X if it satisfies the following conditions:

(i) $d(x,y) \geq 0, \ \forall x,y \in X$

(ii) $d(x,y) = 0 \Leftrightarrow x = y, \ x,y \in X$

(iii) $d(x,y) = d(y,x), \ \forall x,y \in X$. (symmetry)

(iv) $d(x,y) \leq d(x,z) + d(z,y), \ \forall x,y,z \in X$ (triangle inequality)

The ordered pair (X, d) is called a metric space. If there is no confusion likely to occur we, sometimes, denote the metric space (X,d) by X.

Note: When we say that d is a metric on X or (X, d) is a metric space, it is understood throughout that X is a non-empty set.

Remarks 1: The triangle inequality may be interpreted as "the length of one side of a triangle can not exceed the sum of the lengths of the other two sides". Equivalently, the distance from x to y via any intermediate point z can not be shorter than the direct distance from x to y.

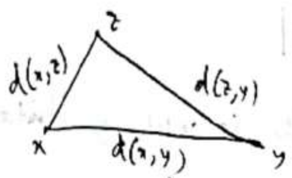

Figure 1.1

2. The triangle inequality can be generalized for any number of additional points $z_1, z_2, \ldots, z_n$ in X, i.e.,

$$d(x,y) \leq d(x,z_1) + d(z_1,z_2) + \ldots + d(z_n,y)$$

1.1.2 Examples

1. Let $X = \mathbb{R}$, the set of all real numbers. For $x,y \in X$, define $d(x,y) = |x-y|$. Then (X,d) is a metric space. This is called the metric space $\mathbb{R}$ with the usual metric and we denote it by $\mathbb{R}_u$.

2. Let $X = \mathbb{C}$, the set of all complex numbers. For $x,y \in X$, define $d(x,y) = |x-y|$. Then (X,d) is a metric space. This is called the metric space $\mathbb{C}$ with the usual metric and we denote it by $\mathbb{C}_u$.

Note: If there is no confusion likely to occur, we may simply write $\mathbb{R}$ (respectively, $\mathbb{C}$) in place of $\mathbb{R}_u$ (respectively, $\mathbb{C}_u$)

3. Let X be an arbitrary non-empty set. For $x,y \in X$, define d by

$$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

Then (X,d) is a metric space. The metric d is called the discrete metric and the space (X,d) is called the discrete metric space and is denoted by X_d.

4. Let $X = \mathbb{Q}$, the set of all rational numbers. For $x,y \in X$, define $d(x,y) = |x-y|$

Then (X,d) is a metric space.

5. Let $X = [0,1)$. For $x, y \in X$, define

$$d(x,y) = |x-y|$$

Then (X, d) is a metric space.

6. Let $X = \mathbb{R}^2$, the set of all points in the coordinate planes. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in X, define

(i) $d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$

(ii) $d_1(x,y) = \max\{|x_1-y_1|, |x_2-y_2|\}$

(iii) $d_2(x,y) = |x_1-y_1| + |x_2-y_2|$

Then each of the spaces (X,d), (X,d_1) and (X,d_2) is a metric space.

7. Let $X = \mathbb{R}^2$. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in X, define

$$d(x,y) = \begin{cases} |x_1 - y_1|, & x_2 = y_2 \\ |x_1| + |y_1| + |x_2 - y_2|, & x_2 \neq y_2 \end{cases}$$

Then (X, d) is a metric space.

8. Let $X = \mathbb{R}^n$, the set of all ordered n-tuples of real numbers. For $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in X, define

$$d(x,y) = \left\{ \sum_{i=1}^{n} (x_i - y_i)^2 \right\}^{1/2}$$

Then (X, d) is a metric space.

9. Let $X = \mathbb{C}^n$, the set of all ordered n-tuples of complex numbers. For $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in X, define

$$d(x,y) = \left\{ \sum_{i=1}^{n} |x_i - y_i|^2 \right\}^{1/2}$$

Then (X, d) is a metric space.

Note: The metric space in Example 8 is called the Euclidean n-space