Proof: The equivalences of (a)⇔(b) and (a)⇔(c) follow trivially in view of theorem 5.1.6 and 5.1.8. It remains only to establish the equivalence of (d) to any of the equivalent statements (a), (b) or (c). We prove (c)⇔(d).

Assume first that (d) be true. Let A be a proper subset of X which is both open and closed. Take $B = X - A$. Clearly $X = A \cup B$ and A, B are disjoint, non-empty, open sets. This verifies (c).

Conversely if (c) be true, then $X = A \cup B$, where A and B are disjoint, non-empty, open sets. Note that $A = X - B$ and as such A is a proper subset of X which is both open and closed.

This completes the proof of the theorem.

5.2.3 Definition: Let $(X, d)$ be a metric space and $Y \subset X$.

(a) The subspace $(Y, d_Y)$ is said to be disconnected if it is disconnected as a metric space in its own right.

(b) The subset Y of X is said to be disconnected if it is disconnected as a subspace of $(X, d)$

(c) A subspace (subset) is said to be connected if it is not disconnected.

5.2.4 Theorem: Let $(X, d)$ be a metric space and $Y \subset X$. Then Y is disconnected if and only if $\exists$ non-empty sets A and B such that

(i) $\bar{A}^X \cap B = \phi$, $A \cap \bar{B}^X = \phi$

(ii) $A \cup B = Y$

where $\bar{A}^X$ stands for the closure of A in X etc.

Proof: First assume that Y is disconnected. By the definition, the metric space $(Y, d_Y)$ is disconnected and so $\exists$ non-empty sets $A, B \subset Y$ such that $\bar{A}^Y \cap B = \phi$, $A \cap \bar{B}^Y = \phi$

and $A \cup B = Y$

Since $\bar{A}^Y = \bar{A}^X \cap Y$, it follows that

$$\bar{A}^Y \cap B = (\bar{A}^X \cap Y) \cap B = \bar{A}^X \cap B$$

$\Rightarrow \bar{A}^X \cap B = \phi$. Similarly, we can show that $A \cap \bar{B}^X = \phi$. This proves one part of the theorem. The converse part is obvious in view of the fact that $\bar{A}^X \supset \bar{A}^Y$ and $\bar{B}^X \supset \bar{B}^Y$.

Note: Theorem 5.2.4 indicates that any two set $A, B \subset Y \subset X$ are separated with respect to $(Y, d_Y)$ if and only if there are so with respect to $(X, d)$

5.2.5 Theorem: Let $(X, d)$ be a metric space and $Y \subset X$. The following statements are equivalent:

(a) Y is disconnected

(b) $Y \subset A \cup B$, where A and B are disjoint open sets such that $Y \cap A$ and $Y \cap B$ are non-empty.

(c) $Y \subset A \cup B$, where A and B are disjoint closed set such that $Y \cap A$ and $Y \cap B$ are non-empty.

(d) $\exists$ a non-empty, proper subset of Y which is both open and closed in $(Y, d_Y)$.

Proof: Prove it.

5.2.6 Theorem : Let $(X, d)$ be a metric space and $Y \subset X$ be connected. Then

(a) If $Y \subset A \cup B$, where $A$ and $B$ are separated sets in $X$, then either $Y \subset A$ or $Y \subset B$.

(b) If $Z \subset X$ be such that $Y \subset Z \subset \bar{Y}$, then $Z$ is connected. In particular $\bar{Y}$ is connected.

Proof (a) Since $A$ and $B$ are separated sets and $Y \cap A \subseteq A$, $Y \cap B \subset B$ it follows by Theorem 5.1.3, that $Y \cap A$ and $Y \cap B$ are separated sets. Moreover, note that

$$(Y \cap A) \cup (Y \cap B) = Y \cap (A \cup B) = Y$$

Thus $Y$ has been expressed as the union of two non-empty separated sets. This verifies $Y$ is ~~connec~~ disconnected which is a contradiction. Hence the result follows.

(b) Let, if possible, $Z$ be disconnected. Then by Theorem 5.2.5, $Z \subset P \cup Q$, where $P$ and $Q$ are disjoint, non-empty, open sets such that $Z \cap P \neq \phi$ and $Z \cap Q \neq \phi$. It is given that $Y \subset Z$. Therefore $Y \subset P \cup Q$. Since $Y$ is connected and $P, Q$ are separated sets, by part (a), either $Y \subset P$ or $Y \subset Q$. We assume, for the sake of definiteness, that $Y \subset P$. Then $Y \cap Q = \phi$. Since $Q$ is open $\bar{Y} \cap Q = \phi$ and since $Z \subset \bar{Y}$. It follows that $Z \cap Q = \phi$. This is a contradiction. Hence $Z$ is connected.

5.2.7 Theorem : Let $(X, d)$ be a metric space and let $\{A_\alpha : \alpha \in I\}$ be a family of connected sets in $X$. such that $\bigcap_{\alpha \in I} A_\alpha \neq \phi$. Then $\bigcup_{\alpha \in I} A_\alpha$ is connected.

Proof: Let $\bigcup_{\alpha \in I} A_\alpha$ be not connected. Then $\exists$ non-empty separated sets $P$ and $Q$ such that $\bigcup_{\alpha \in I} A_\alpha = P \cup Q$

Since $P$ and $Q$ are nonempty $\exists$ $\alpha, \beta \in I$ with $\alpha \neq \beta$ such that $A_\alpha \subset P$ and $A_\beta \subset Q$

Then $A_\alpha$ and $A_\beta$ are separated sets (Theorem 5.1.3) and as such $A_\alpha \cap A_\beta = \phi$. This is a contradiction to the hypothesis that $\bigcap_{\alpha \in I} A_\alpha \neq \phi$.

Hence $\bigcup_{\alpha \in I} A_\alpha$ is connected.

5.2.8 Theorem: Let $(X, d)$ be a metric space and $A \subset X$. If every pair of elements in $A$ lies in a connected subset of $A$, then $A$ is connected.

Proof: Let, if possible, $A$ be disconnected. Then, $\exists$ non-empty separated ~~subsets~~ sets $P$ and $Q$ such that $A = P \cup Q$. Let $x \in P$ and $y \in Q$ (This is possible since $P$ and $Q$ are non-empty). Then $x, y \in A$. By the hypothesis $\exists$ a connected set $A_{xy} \subset A$ such that $x, y \in A_{xy}$. As such either $A_{xy} \subset P$ or $A_{xy} \subset Q$ (Theorem 5.2.6(a)). Suppose $A_{xy} \subset P$. Then $y \in A_{xy} \subset P$. On the other hand $y \notin P$ since $P$ and $Q$ are separated. This is a contradiction.

Hence $A$ is connected.

5.2.8. Definition: Let $(X, d)$ be a metric space and $A \subset X$. The set $A$ is said to be totally disconnected if ~~no~~ two points of $A$ lie in a connected subset of $A$.