5.3 Connected subsets of $\mathbb{R}$

6.4.1 Theorem: A subset $A \subset \mathbb{R}$ in the usual metric space $\mathbb{R}_u$ is connected if and only if it is an interval.

Proof: Assume that A is connected. Let, if possible, A be not an interval. Then $\exists$ real numbers x, y, z such that $x < y < z$ with $x, z \in A$ and $y \notin A$. This shows that $A = (A \cap (-\infty, y)) \cup (A \cap (y, \infty))$

and $(A \cap (-\infty, y)) \cap (A \cap (y, \infty)) = \phi$

Moreover the set $A \cap (-\infty, y)$ and $A \cap (y, \infty)$ are nonempty, since $x \in A \cap (-\infty, y)$ and $z \in A \cap (y, \infty)$ and open. Thus A is the union of two non-empty disjoint open sets and so, A is disconnected, a contradiction. Hence A is an interval.

Conversely, let A be an interval. Suppose A is disconnected. Then $A = G \cup H$, where G and H are non-empty, disjoint, closed sets in A. Let $x \in G$ and $z \in H$. Then $x \neq z$ since G and H are disjoint. We may assume that $x < z$. Since, A is an interval, $[x, z] \subset A$ and a point in $[x, z]$ is either in G or in H.

Define y by $y = \sup\{t \in G : x \leq t \leq z\}$

It is clear that $x \leq y \leq z$ and so $y \in A$. But G being ~~a set~~ closed in A, by the definition of y, $y + \varepsilon \in H$, for each $\varepsilon > 0$ such that $y + \varepsilon \leq z$. Furthermore, since H is closed in A, $y \in A$.

Then $y \in G$ and $y \in H$, a contradiction. Hence A

7. Fixed point theorem and their applications.

7.1 Definition : Let (X, d) be a metric space and $T: X \to X$ be a mapping. The point $x \in X$ is called a fixed point of T if $T(x) = x$.

7.2 Definition : Let (X, d) be a metric space. A mapping $T: X \to X$ is called a contraction of X if $\exists$ a real number k with $0 \leq k \leq 1$ such that $d(T(x), T(y)) \leq k\, d(x, y)$ $\forall\, x, y \in X$, $x \neq y$. The number k is called Lipschitz constant of T.

7.2.1 Theorem (Banach's Fixed Point Theorem)
Let (X, d) be a complete metric space and let $T: X \to X$ be a contraction on X. Then T has a unique fixed point in X.

Proof: Let $k \in [0, 1]$ be the Lipschitz constant such that $d(T(x), T(y)) \leq k\, d(x, y)$, $\forall\, x, y \in X$

We prove the theorem in various steps.

Step (i) we construct a sequence $\{x_n\} \subset X$ as follows:
Take any point $x_0 \in X$ and inductively construct the sequence $\{x_n\}$ of points in X as:

$$
\begin{aligned}
x_1 &= T(x_0) \\
x_2 &= T(x_1) = T^2(x_0) \\
&\cdots\cdots \\
x_n &= T(x_{n-1}) = T^n x_0 \\
&\cdots\cdots
\end{aligned}
$$

Clearly $\{x_n\}$ is the sequence of images of x_0 under repeated application of T.

Step (ii) $\{x_n\}$ is a Cauchy sequence in X

Let $m<n$. Then $d(x_m, x_n) = d(T^m(x_0), T^n(x_0))$

$$\leq k\, d(T^{m-1}(x_0), T^{n-1}(x_0))$$

$$\cdots\cdots$$

$$\cdots\cdots$$

$$\leq k^m d(x_0, T^{n-m}(x_0))$$

$$\leq k^m \left[d(x_0, T(x_0)) + d(T(x_0), T^2(x_0)) + \cdots + d(T^{n-m-1}x_0, T^{n-m}(x_0)) \right]$$

(by triangle inequality)

$$\leq k^m \left[d(x_0, T(x_0)) + k\, d(x_0, T(x_0)) + \cdots + k^{n-m-1} d(x_0, T(x_0)) \right]$$

$$= k^m \left[1 + k + k^2 + \cdots + k^{n-m-1} \right] d(x_0, T(x_0))$$

$$\leq \frac{k^m}{1-k} d(x_0, T(x_0)), \quad (0 \leq k < 1)$$

$\to 0$ as $m \to \infty$ (and hence $n \to \infty$). Hence

$\{x_n\}$ is a Cauchy sequence

Step (iii) Since X is complete and $\{x_n\}$ is a Cauchy sequence in X, $\exists\ x \in X$ such that $x_n \to x$.

Step (iv) x is a fixed point of T

We have $d(x, T(x)) \leq d(x, x_n) + d(x_n, T(x))$ (by triangle inequality)

$$\leq d(x, x_n) + k\, d(x_{n-1}, x) \quad (\because x_n = T(x_{n-1}))$$

$$\to 0 \quad \text{as } n \to \infty$$

$$\Rightarrow \quad d(x, T(x)) = 0$$

$$\Rightarrow \quad T(x) = x$$

Hence x is a fixed point of T.

Thus the existence of a fixed point is established.

In the last step, we verify the uniqueness of such a fixed point.

Step (v) x is a unique fixed point of T.

Let, if possible, x and y be two fixed points of T.

Then $T(x) = x$, $T(y) = y$

Now, note that $d(x,y) = d(T(x), T(y)) \leq k\, d(x,y)$

$$\Rightarrow d(x,y) = 0$$

$$\Rightarrow x = y$$

This completes the proof of the theorem.

7.3 Application of Banach Fixed Point Theorem

7.2.1 Definition (Solution of a Differential equation in an interval): Let f be a real valued function on a non-empty subset D of the Euclidean plane $\mathbb{R}^2$. A real valued function g on an interval I said to be a solution of the differential equation $\frac{dy}{dx} = f(x,y)$ on the interval I, if

(i) $(x, g(x)) \in D \quad \forall x \in I$

(ii) $g'(x)$ exists and equals $f(x, g(x))$, for each $x \in I$.

7.2.2. Theorem (Picard's Theorem): Let D be a non-empty open subset of the Euclidean plane $\mathbb{R}^2$, $f: D \to \mathbb{R}$ be a continuous map which satisfies Lipschitz condition with respect to the second variable, i.e.,

$|f(x,y_1) - f(x,y_2)| \leq k|y_1 - y_2| \quad \cdots (1)$ for all $(x,y_1), (x,y_2) \in D$ and for some $k > 0$, and let $(x_0, y_0) \in D$

Then the differential equation $\frac{dy}{dx} = f(x,y) \quad \cdots (2)$ has a unique solution $y = g(x)$ in the interval $[x_0 - c, x_0 + c]$, for a suitable $c > 0$, such that the boundary