condition $g(x_0) = y_0$ is satisfied.

Proof: In order to prove the theorem we first establish that whenever g is a real valued function on some closed interval I with values in a closed interval $\tilde{I}$ such that $I \times \tilde{I} \subseteq D$, and $x_0 \in I$, then $y = g(x)$ is a solution of the differential equation (2) on I with $g(x_0) = y_0$ if and only if $y = g(x)$ is a solution on I of the integral equation $y = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt$ . . . (3)

~~Indeed, suppose that y = g(x) is a solution of the differential equation (2) on I and g(x₀) = y₀. Then g is differentiable on I and g′(t) = f(t, g(t)), ∀ t ∈ I. Hence g(x) − y₀ = g(x) − g(x₀) = ∫ g′(t) dt = ∫ f(t, g(t)) dt ∀ x ∈ I. Since the integrand on the right hand side is a continuous function of t (verify), g is differentiable on I and g′(x) = f(x, g(x)) ∀ x ∈ I (by known result of Integral calculus). This shows that y = g(x) is a solution of the differential equation and g′(t) = f(t, g(t))~~

Indeed, suppose that $y = g(x)$ is a solution of the differential equation (2) on I, and $g(x_0) = y_0$. Then g is differentiable on I, and $g'(t) = f(t, g(t))$, $\forall t \in I$. Hence $g(x) - y_0 = g(x) - g(x_0) = \int_{x_0}^{x} g'(t)\,dt = \int_{x_0}^{x} f(t, g(t))\,dt$, $\forall x \in I$ which shows that $y = g(x)$ is a solution

Conversely, assuming that $y = g(x)$ is a solution of (3) on I, we get $g(x) = y_0 + \int_{x_0}^{x} f(t, g(t))\,dt$, $\forall x \in I$. Since the integrand on the right side is a continuous function of t (verify), g is differentiable on I and $g'(x) = f(x, g(x))$, $\forall x \in I$ (by known result of integral calculus). This shows that $y = g(x)$ is a solution of the differential equation (2). Obviously $g(x_0) = y_0$ also holds.

We now proceed to prove the theorem itself.

Since the function f is continuous on D, we have $|f(x,y)| \leq M$ (for some $M > 0$) in some open ball $G \subseteq D$ such that $(x_0, y_0) \in G$. Now, we can choose a real number $c > 0$ such that $[x_0 - c, x_0 + c] \times [y_0 - Mc, y_0 + Mc] \subseteq G$ and $kc < 1$.

Let E denote the set of all continuous mappings on the closed interval $[x_0 - c, x_0 + c]$ into the closed interval $[y_0 - Mc, y_0 + Mc]$. Then E forms a complete metric space with respect to the uniform metric $d(h_1, h_2) = \sup\{|h_1(x) - h_2(x)| : x \in [x_0 - c, x_0 + c]\}$, where $h_1, h_2 \in E$. Let us consider the map

$T : E \to E$ given by :

For $g \in E$, $T(g) = h$, where $h(x) = y_0 + \int_{x_0}^{x} f(t, g(t))\,dt$, $\forall x \in [x_0 - c, x_0 + c]$

That $T(g) \in E$ follows from the following considerations: $x \in [x_0 - c, x_0 + c] \Rightarrow |h(x) - y_0| =$

$= \left| \int_{x_0}^{x} f(t, g(t))\,dt \right| \leq M|x - x_0| \leq Mc \Rightarrow$

$h(x) \in [y_0 - Mc, y_0 + Mc]$. Also, clearly h is continuous. Thus $T(g) \in E$, $\forall g \in E$

Now, for $g_1, g_2 \in E$ $T(g_i) = h_i$ for $i = 1, 2$. Then for all $x \in [x_0 - c, x_0 + c]$ with $x \geq x_0$,

$$|h_1(x) - h_2(x)| = \left| \int_{x_0}^{x} \{ f(t, g_1(t)) - f(t, g_2(t)) \} dt \right|$$

$$\leq k \int_{x_0}^{x} |g_1(t) - g_2(t)|\,dt \leq k \sup_{t \in [x_0 - c, x_0 + c]} |g_1(t) - g_2(t)| \cdot c$$

$= kc \cdot d(g_1, g_2)$, the same inequality is valid for $x \leq x_0$

Hence $\sup_{x \in [x_0 - c, x_0 + c]} |h_1(x) - h_2(x)| \leq kc \cdot d(g_1, g_2)$

i.e., $d(h_1, h_2) \leq kc \cdot d(g_1, g_2)$ with $kc < 1$

i.e., $d(T(g_1), T(g_2)) \leq kc\, d(g_1, g_2)$ with $kc < 1$

So, $T : E \to E$ is a contraction mapping. Since E is a complete metric space, By Banach's fixed point theorem, T has a unique fixed point g (say) $\in E$ so that $g = T(g)$. Then $y = g(x)$ is a solution of the integral equation (3). Then by what was observed at the outset, the differential equation (2) has a solution $y = g(x)$ for which the given boundary condition $y_0 = g(x_0)$

also holds.

Let g_1 be another solution of the given differential equation with $g_1(x_0) = y_0$. Then $g_1'(x) = f(x, g(x))$, $\forall x \in [x_0 - c, x_0 + c]$ and $g_1(x_0) = y_0$

Now g_1 is a solution of the integral equation (3) and hence $g_1(x) = y_0 + \int_{x_0}^{x} f(t, g_1(t))\, dt$

$= T(g_1)(x)\ \forall x \in [x_0 - c, x_0 + c]$, i.e., $T(g_1) = g_1$, proving that g_1 is a fixed point of T.

By uniqueness of the fixed point of T, $g = g_1$. Hence the solution of the differential equation (2), satisfying the given condition is unique. This completes the proof.