and is denoted by $\mathbb{R}^n$ while the metric space in Example 9 is called the unitary n-space (or Complex Euclidean n-space) and is denoted by $\mathbb{C}^n$.

10. Let $X = \mathbb{K}^n$ ($\mathbb{R}^n$ or $\mathbb{C}^n$). Define

(i) $d_1(x,y) = \max\limits_{1 \le i \le n} |x_i - y_i|$

(ii) $d_2(x,y) = \sum\limits_{i=1}^{n} |x_i - y_i|$ for $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$

Then each of the spaces (X, d_1) and (X, d_2) is a metric space.

11. Let $X = \mathbb{K}^n$ ($\mathbb{R}^n$ or $\mathbb{C}^n$). Define

$$d_p(x,y) = \left\{ \sum_{i=1}^{n} |x_i - y_i|^p \right\}^{1/p}, \ 1 \le p < \infty; \ x = (x_1, x_2, \ldots, x_n) \text{ and } y = (y_1, y_2, \ldots, y_n)$$

Then (X, d_p) is a metric space. We denote the metric space $(\mathbb{K}^n, d_p)$ by l_n^p.

Note: l_p^2 is the Euclidean ~~space~~ metric n-space $\mathbb{R}^n$ or Unitary n-space $\mathbb{C}^n$ according as $X = \mathbb{R}^n$ or $\mathbb{C}^n$

Let us now generalize $\mathbb{R}^n$ (and $\mathbb{C}^n$), in Example 11 above to 'infinite tuples', which, in fact, are the sequences in $\mathbb{K}$. While doing so, we shall be putting suitable restrictions on the sequences to be considered so as to make them meaningful.

12. Let $1 \le p < \infty$. Consider the set of sequences $\{x_n\}$ in $\mathbb{K}$ such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty .$$

Denote this set by l^p. For $x = \{x_n\}$ and $y = \{y_n\}$ in l^p, define $d_p(x,y) = \left\{ \sum\limits_{n=1}^{\infty} |x_n - y_n|^p \right\}^{1/p}$. Then (X, d_p) is a metric space where $X = l^p$

13. Let l^∞ be the set of all bounded sequences in $\mathbb{K}$; i.e.,

$$l^\infty = \left\{ \{x_n\} : \{x_n\} \text{ is a sequence in } \mathbb{K} \text{ and } \sup_{1 \le n < \infty} |x_n| < \infty \right\}$$

For $x = \{x_n\}$ and $y = \{y_n\}$ in l^∞, define

$$d_\infty(x, y) = \sup_{1 \le n < \infty} |x_n - y_n|$$

Then d_∞ defines a metric on l^∞

14. Let $B[a,b]$ be the set of all real valued functions defined and bounded on $[a,b]$. For $x, y \in B[a,b]$, define

$$d_\infty(x,y) = \cancel{\sup_{t\in[a,b]} |x(t)-y(t)|} \sup_{t \in [a,b]} |x(t) - y(t)|$$

Then $(B[a,b], d_\infty)$ is a metric space.

15. Let $C[a,b]$ be the set of all real valued continuous functions defined on $[a,b]$. For $x, y \in C[a,b]$, define

$$d_\infty(x,y) = \max_{t \in [a,b]} |x(t) - y(t)|$$

Then $(C[a,b], d_\infty)$ is a metric space.

Note: The metric d_∞ defined on $B[a,b]$ and $C[a,b]$, in Examples 14 and 15, is called the uniform metric. This metric, in fact, corresponds to the uniform convergence of functions in the space $C[a,b]$.

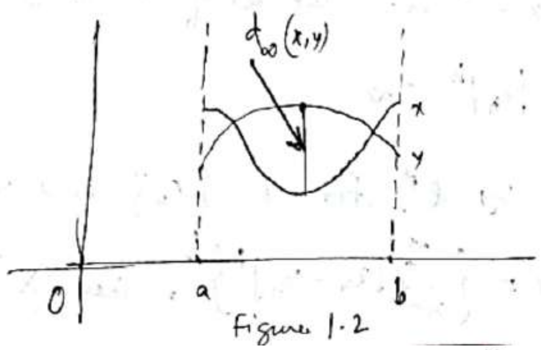


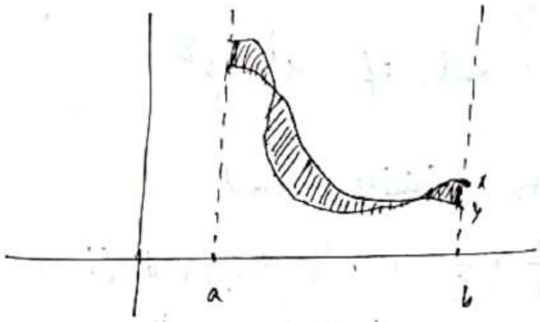
Figure 1.2

16. For $x, y \in C[a,b]$, define

$$d_1(x,y) = \int_a^b |x(t) - y(t)|\, dt$$

the integral on the right being taken in the sense of Riemann which is possible since the functions x and y are continuous on $[a,b]$. Then $(C[a,b], d_1)$ is a metric space.

Note: $d_1(x,y)$ represents the absolute area between the functions x and y as a measure of the distance between two function



17. For $x, y \in C[a,b]$, define

$$d_p(x,y) = \left\{ \int_a^b |x(t) - y(t)|^p \right\}^{1/p}, \quad 1 \leq p < \infty$$

Then $(C[a,b], d_p)$ is a metric space

18. Let $X = \mathbb{R}$, For $x, y \in X$, define

$$d(x,y) = |x^2 - y^2|$$

Then (X, d) is not a metric space

19. Let $X = \mathbb{R}$. For $x, y \in X$, define

$$d(x,y) = |\sin(x-y)|$$

Then (X, d) is not a metric.

Exercises: Prove all the results stated in Example 1 to Example 19

To prove the above results, you can use the following inequalities: Here $\mathbb{K}$ implies $\mathbb{R}$ or $\mathbb{C}$

Inequalities

1. The triangle inequality

Let $\alpha, \beta \in \mathbb{K}$. Then $|\alpha+\beta| \le |\alpha| + |\beta|$

2. Let $\alpha, \beta \in \mathbb{K}$. Then

$$\frac{|\alpha+\beta|}{1+|\alpha+\beta|} \le \frac{|\alpha|}{1+|\alpha|} + \frac{|\beta|}{1+|\beta|}$$

3. Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\alpha, \beta \in \mathbb{K}$, then

$$\alpha\beta \le \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$

with equality if and only if $\alpha^p = \beta^q$

4. Holder's Inequality (finite form)

Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\alpha_i, \beta_i \in \mathbb{K}$ $(i=1,2,\ldots,n)$

then $$\sum_{i=1}^{n} |\alpha_i \beta_i| \le \left(\sum_{i=1}^{n} |\alpha_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |\beta_i|^q\right)^{1/q}.$$

Also $$\sum_{i=1}^{n} |\alpha_i \beta_i| \le \left(\sum_{i=1}^{n} |\alpha_i|\right) \cdot \max_{1\le i\le n} |\beta_i|$$

5. Cauchy-Schwarz Inequality (finite form)

Note the inequality 4 for the case when $p = q = 2$

$$\sum_{i=1}^{n} |\alpha_i \beta_i| \le \left(\sum_{i=1}^{n} |\alpha_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} |\beta_i|^2\right)^{1/2}.$$

6. Minkowski's inequality (finite form)

Let $1 \le p < \infty$. If $\alpha_i, \beta_i \in \mathbb{K}$ $(i=1,2,\ldots,n)$ then

$$\left(\sum_{i=1}^{n} |\alpha_i+\beta_i|^p\right)^{1/p} \le \left(\sum_{i=1}^{n} |\alpha_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |\beta_i|^p\right)^{1/p}.$$