7. Let $0 \le p \le 1$. If $\alpha_i, \beta_i \in \mathbb{K}$ $(i=1,2,\dots,n)$, then

$$\sum_{i=1}^{n} |\alpha_i + \beta_i|^p \le \sum_{i=1}^{n} |\alpha_i|^p + \sum_{i=1}^{n} |\beta_i|^p$$

8. Holder's inequality (Infinite form)

Let $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $(\alpha_1, \alpha_2, \dots) \in l^p$ and $(\beta_1, \beta_2, \dots) \in l^q$, i.e.,

$$\sum_{i=1}^{\infty} |\alpha_i|^p < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} |\beta_i|^q < \infty$$

Then $$\sum_{i=1}^{\infty} |\alpha_i \beta_i| \le \left(\sum_{i=1}^{\infty} |\alpha_i|^p\right)^{1/p} \left(\sum_{i=1}^{\infty} |\beta_i|^q\right)^{1/q}$$

9. Cauchy Schwarz Inequality (Infinite form)

write the inequality (8) when $p = q = 2$

$$\sum_{i=1}^{\infty} |\alpha_i \beta_i| \le \left(\sum_{i=1}^{\infty} |\alpha_i|^2\right)^{1/2} \left(\sum_{i=1}^{\infty} |\beta_i|^2\right)^{1/2}$$

10. Minkowski's inequality (Infinite form)

Let $1 \le p < \infty$ If $(\alpha_1, \alpha_2, \dots), (\beta_1, \beta_2, \dots) \in l^p$: i.e.,

$$\sum_{i=1}^{\infty} |\alpha_i|^p < \infty, \quad \sum_{i=1}^{\infty} |\beta_i|^p < \infty$$

Then $$\left(\sum_{i=1}^{\infty} |\alpha_i + \beta_i|^p\right)^{1/p} \le \left(\sum_{i=1}^{\infty} |\alpha_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |\beta_i|^p\right)^{1/p}$$

1.2 OPEN BALL AND CLOSED BALL (~~[illegible]~~ OR SPHERE)

1.2.1 Definition: Let (X, d) be a metric space. Let $x \in X$ and $r > 0$ be a real number. The open ball with centre x and radius r, denoted by $B_r(x)$, is the subset of X

given by $B_r(x) = \{y \in X : d(x,y) < r\}$

Note: An open ball is always non-empty since it contains its centre at least.

1.2.2 Examples :

1. In the usual metric space $\mathbb{R}_u$, the open ball $B_r(x_0)$ is the open interval (x_0-r, x_0+r), $x_0 \in \mathbb{R}$ and $r>0$

Remark: Every open ball in the usual metric space $\mathbb{R}_u$ is an open interval. But the converse is not true; for instance $(-\infty, \infty)$ is an open interval in $\mathbb{R}$ but not an open ball.

2. In the usual metric space $\mathbb{C}_u$ the open ball $B_r(z_0)$ is the circular disc $|z-z_0|<r$, $z_0 \in \mathbb{C}$ and $r>0$.

3. Let x_0 be any point in the discrete metric space X_d (Example 1.1.2 (3)). Then

$$B_r(x_0) = \begin{cases} \{x_0\}, & 0<r\leq 1 \\ X, & r>1 \end{cases}$$

4. In the metric space in Example 2.1.2 (5),

$$B_r(0) = \begin{cases} [0,r), & r\leq 1 \\ [0,1), & r>1 \end{cases}$$

5. In the metric space $(\mathbb{R}^2, d)$ of Example 1.1.2 (6), the unit ball centred at the origin is given by

$$B_1((0,0)) = \{(x_1,x_2) : x_1^2+x_2^2<1\}$$

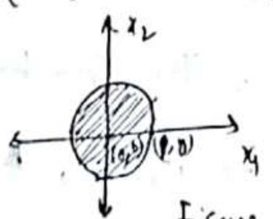

Figure 1.4

6. In the metric space $(\mathbb{R}^2, d_1)$ of Example 1.1.2(6), the unit ball centred at origin is given by

$$B_1((0,0)) = \{(x_1,x_2) : |x_1|<1,\ |x_2|<1\}$$

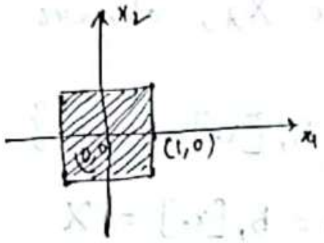

Figure 1.5

7. In the metric space $(\mathbb{R}^2, d_2)$ of Example 1.1.2(6), the unit ball centred at the origin is given by

$$B_1((0,0)) = \{(x_1,x_2) : |x_1|+|x_2| < 1\}$$

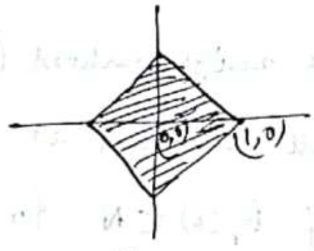

Figure 1.6

1.2.3 Definition : Let (X, d) be a metric space. Let $x \in X$ and $r>0$. The closed ball (or sphere) with centre x and radius r, denoted by $B_r[x] = \{y : d(x,y) \le r\}$

1.2.4 Examples

1. In the usual metric space $\mathbb{R}_u$, the closed ball $B_r[x_0]$ is the closed interval $[x_0-r,\ x_0+r]$.

2. In the usual metric space $\mathbb{C}_u$, the closed ball $B_r[z_0]$ is the closed disc $|z-z_0| \le r$, $z_0 \in \mathbb{C}$ and $r>0$.

3. In the discrete metric space X_d (Example 1.1.2(3)), the

closed ball $B_r[x_0]$ is given by

$$B_r[x_0] = \begin{cases} \{x_0\}, & 0<r<1 \\ X, & r \geq 1 \end{cases}$$

Remarks : In the metric space X_d, observe that

(i) For $0<r<1$ $B_r(x_0) = B_r[x_0] = \{x_0\}$.

(ii) For $r>1$ $B_r(x_0) = B_r[x_0] = X$

(iii) For $r=1$, $B_r(x_0) = \{x_0\}$, $B_r[x_0] = X$

1.3 NEIGHBOURHOODS

1.3.1 Definition : Let (X, d) be a metric space. A set $N \subset X$ is said to be a neighbourhood (nbd. in short) of x if $\exists$ an open ball centred at x and is contained in N, i.e., if $B_r(x) \subset N$ for some $r>0$.

1.3.2 Examples

1. The open interval (a, b) is a nbd of each of its points in the usual metric space $\mathbb{R}_u$

2. The $\mathbb{R}$ of real numbers is a nbd of each of its points in $\mathbb{R}_u$

3. The closed intervals $[a, b]$ is a nbd of each point of (a, b) but not a nbd of the end points a and b in $\mathbb{R}_u$

4. The set $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Q}$ is not a nbd of any of its points in $\mathbb{R}_u$

5. In a discrete metric space X_d, a subset $Y \subset X$