is a nbd of each of its points.

1.3.3 Theorem : Let (X, d) be a metric space. A set $N \subset X$ is a nbd of a point $p \in X$ if and only if $\exists$ an open ball $B_r(x)$ such that $p \in B_r(x) \subset N$

Proof: First we assume that N is a nbd of a point $p \in X$. Then $\exists$ an $r > 0$ such that $p \in B_r(p) \subset N$. This proves that $\exists$ an open ball $B_r(p)$ containing p and contained in N.

Conversely, assume that $p \in N$ and $\exists$ an open ball $B_r(x)$ such that $p \in B_r(x) \subset N$. We shall prove that $\exists$ an open ball centred at p and contained in N. Now

$p \in B_r(x) \Rightarrow d(x, p) < r$. Let $r_1 = r - d(x, p)$.

Then $r_1 > 0$. Let $y \in B_{r_1}(p)$. Then $d(y, p) < r_1$. Now, we have $d(y, p) < r_1 \Rightarrow d(y, p) < r - d(x, p)$

$$\Rightarrow d(x, p) + d(y, p) < r$$

$$\Rightarrow d(x, p) + d(p, y) < r \quad [\because d(y, p) = d(p, y)]$$

$$\Rightarrow d(x, y) < r \quad [\because d(x, y) \leq d(x, p) + d(p, y)]$$

$$\Rightarrow y \in B_r(x)$$

So, $B_{r_1}(p) \subset B_r(x) \subset N$. This verifies that N is a nbd of p.

2.3.4 Theorem : Let (X, d) be a metric space and $x \in X$. Let $\mathbb{N}_x$ be the collection of all nbds. of x.

Then : (a) $M, N \in \mathbb{N}_x \Rightarrow M \cap N \in \mathbb{N}_x$

(b) $N \in \mathbb{N}_x$ and $M \supset N \Rightarrow M \in \mathbb{N}_x$

Proof: (a) We have

$M, N \in \mathbb{N}_x \Rightarrow \exists\, r_1, r_2 > 0$ such that $B_{r_1}(x) \subset M$ and $B_{r_2}(x) \subset N$

$\Rightarrow B_r(x) \subset M$ and $B_r(x) \subset N$, where $r = \min\{r_1, r_2\}$

$\Rightarrow B_r(x) \subset M \cap N$

$\Rightarrow M \cap N$ is a nbd of x

$\Rightarrow M \cap N \in \mathbb{N}_x$

(b) We have

$N \in \mathbb{N}_x \Rightarrow \exists$ an $r > 0$ such that $S_r(x) \subset N$

$\Rightarrow S_r(x) \subset M \quad (\because M \supset N)$

$\Rightarrow M \in \mathbb{N}_x$.

1.4. Open sets

1.4.1 Definition : Let (X, d) be a metric space. A set $G \subset X$ is said to be an open set if it is a nbd of each of its points.

(Equivalently, a set $G \subset X$ is said to be an open set if for each $x \in G$, $\exists$ an $r > 0$ such that $B_r(x) \subset G$.

1.4.2 Examples

1. In the usual metric space $\mathbb{R}_u$

(a) $\mathbb{R}$ is an open set

(b) $(0,1)$ is an open set

(c) $[0,1)$ is not an open set

(d) The set $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are not open sets.

(e) The set of all irrational numbers is not an open set

(f) $\{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ is not an open set

(g) $\{x\}$, $x \in \mathbb{R}$, is not an open set.

2. In the metric space of Example 1.1.2(5), $[0, \alpha)$, $\alpha \leq 1$, is an open set.

3. In the metric space of Example 1.1.2(4), the set

(a) $\emptyset$ is an open set

(b) $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ is not an open set.

4. In the discrete metric space X_d, every set $G \subset X$ is an open set. In particular, every singleton set in X_d is open.

1.4.3 <u>Theorem</u> : Let (X, d) be a metric space. Then, the empty set $\emptyset$ and the whole space X are open sets.

Proof: In order to prove that $\emptyset$ is an open set, we need to verify that each point of $\emptyset$ is the centre of some open ball contained in $\emptyset$. But $\emptyset$ contains no point. Therefore, the requirement is automatically satisfied.

For the second part, let $x \in X$ be arbitrary. Then $\exists$ an open ball $B_r(x) \subset X$. This is possible since any open ball centred ~~at~~ on a point of X can not go beyond X. Thus X is a nbd. of x.

But $x \in X$ is arbitrary. Therefore X is a nbd of each of its points. Hence X is an open set.

1.4.4. <u>Theorem</u>: Let (X, d) be a metric space. Then, each open ball in X is an open set.

Proof: Let $B_r(x_0) = \{x \in X : d(x, x_0) < r\}$ be an open ball in (X, d). Let $y_0 \in B_r(x_0)$ be arbitrary but fixed. Then $d(x_0, y_0) < r$. Write $r_1 = r - d(x_0, y_0)$

Clearly, $r_1 > 0$. Consider $B_{r_1}(y_0) = \{y \in X : d(y, y_0) < r_1\}$

Let $y \in B_{r_1}(y_0)$ be arbitrary. Then $d(y, y_0) < r_1$

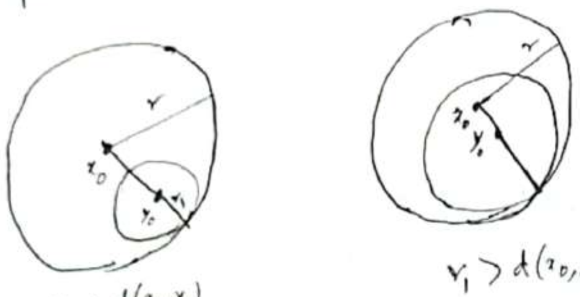
Now $d(x_0, y) \leq d(x_0, y_0) + d(y_0, y)$ (by triangle inequality)

$$< d(x_0, y_0) + r_1 = r$$

$\Rightarrow y \in B_r(x_0)$. Consequently,

$$B_{r_1}(y_0) \subset B_r(x_0)$$

This proves that $B_r(x_0)$ is a nbd of y_0. But $y_0 \in B_r(x_0)$ is arbitrary. So, $B_r(x_0)$ is a nbd of each of its points. Hence $B_r(x_0)$ is an open set



$r_1 < d(x_0, y_0)$ $\qquad$ $r_1 > d(x_0, y_0)$

Figure 1.7

Remark: Converse of the above theorem need not be true. For instance, the interval $(-\infty, \infty)$ in the usual metric space $\mathbb{R}_u$ is an open set while it is not open ball.

1.4.5 <u>Theorem</u> : Let (X, d) be a metric space and $G \subset X$. Then, G is an open set if and only if it is the union of open balls.