Proof: Let G be an open set. Then each point of G is the centre of an open ball contained in G. Clearly, union of all ~~such~~ such open balls is precisely the set G.

Conversely, let G be the union of open balls. Let $\mathcal{C}$ be ~~the~~ the family of these open balls. Let $x \in G$ be arbitrary. Then, x belongs to some open ball, say $B_r(x_0) \in \mathcal{C}$. Since every open ball is an open set, x is the centre ~~of~~ of an open ball $B_{r_1}(x)$ such that

$$B_{r_1}(x) \subset B_r(x_0)$$

But
$$B_r(x_0) \subset G$$

So,
$$B_{r_1}(x) \subset G.$$

This proves that G is a nbd of x. But $x \in G$ is arbitrary. So, G is a nbd of each of its points. Hence G is an open set.

1.4.6 <u>Theorem</u> : Let (X, d) be a metric space. Then

(a) Arbitrary union of open sets in X is open

(b) Finite intersection of open sets in X is open.

Proof: (a) Let $\{G_\alpha : \alpha \in I\}$ be a family ~~of~~ of open sets in X. We shall prove that $\bigcup_{\alpha \in I} G_\alpha$ is open. Since each G_α is open, it is the union of open balls for each $\alpha \in I$. Then $\bigcup_{\alpha \in I} G_\alpha$ is the union of of unions of open balls. Hence by Theorem 1.4.5, it is open.

(b) Let $\{G_i : i = 1, 2, \ldots, n\}$ be the finite family of open sets in X. We shall prove that $\bigcap_{i=1}^{n} G_i$ is open.

Let $x \in \bigcap_{i=1}^{n} G_i$ be arbitrary. Then

$x \in G_i$ for each $i = 1, 2, \ldots, n$

$\Rightarrow \exists$ an $r_i > 0$ such that $B_{r_i}(x) \subset G_i$, $(i = 1, 2, \ldots, n)$
($\because$ Each G_i is open)

Write $r = \min\limits_{1 \le i \le n} r_i$

Then $B_r(x) \subset B_{r_i}(x) \subset G_i$ $(i = 1, 2, \ldots, n)$

$\Rightarrow B_r(x) \subset \bigcap_{i=1}^{n} G_i$

Hence $\bigcap_{i=1}^{n} G_i$ is open.

Remark: Arbitrary intersection of open sets need not be open.

1.4.7 <u>Example</u>: In the usual metric space $\mathbb{R}_u$, consider the family $\{(-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ of open sets. Then

$$\bigcap_{n \in \mathbb{N}} (-\tfrac{1}{n}, \tfrac{1}{n}) = \{0\}$$ which is not an open set.

Remark: Open sets are among the most important concepts in geometry and analysis. One may note how often ideas are defined in terms of open sets and there are results and theorems which give necessary and sufficient conditions for these new ideas in terms of open sets. Furthermore, the basic foundation of topology lies on the concept of open sets.

1.5. CLOSED SETS

1.5.1 Definition : Let (X, d) be a metric space and $A \subset X$. The set A is said to be closed if its complement $X - A$ is open.

1.5.2 Examples

1. In the usual metric space $\mathbb{R}_u$, the set

(a) $A = [1, 2]$ is closed since $\mathbb{R} - A = (-\infty, 1) \cup (2, \infty)$ is open.

(b) $A = (1, 2)$ is not closed since $\mathbb{R} - A = (-\infty, 1] \cup [2, \infty)$ is not open

(c) $A = \mathbb{Q}$ is not closed since $\mathbb{R} - \mathbb{Q}$, the set of all irrational numbers, is not open.

(d) $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ is not closed since
$\mathbb{R} - A = (-\infty, 0] \cup \ldots \cup (\frac{1}{2}, 1) \cup (1, \infty)$ is not open

(e) $A = \mathbb{R}$ is closed since $\mathbb{R} - A = \phi$ is open

2. In a discrete metric space X_d, a subset $Y \subset X$ is closed.

1.5.3 Theorem : Let (X, d) be a metric space. Then, the empty set ϕ and the whole space X are closed sets.

Proof : By Theorem 1.4.3, ϕ and X are open sets. Therefore, their complements X and ϕ are closed sets.

1.5.4 Theorem: Let (X, d) be a metric space. Then, each closed ball in X is a closed set.

Proof: Let $B_r[x]$ be a closed ball in (X, d). It is sufficient to prove that $X - B_r[x]$ is an open set. Let $y \in X - B_r[x]$ be arbitrary. Then $y \notin B_r[x]$ and so, $d(x, y) > r$.

Write $r_1 = d(x,y) - r$. Then $r_1 > 0$. Let $z \in B_{r_1}(y)$. Then $d(y, z) < r_1$

By triangle inequality, we have

$$d(x,y) \leq d(x,z) + d(z,y)$$

$$\Rightarrow \quad d(x,z) \geq d(x,y) - d(y,z) \quad \left(d(y,z) = d(z,y)\right)$$

$$> d(x,y) - r_1 = r$$

$$\Rightarrow \quad z \notin B_r[x]$$

$$\Rightarrow \quad z \in X - B_r[x]$$

Thus $B_{r_1}(y) \subset X - B_r[x]$ and hence $X - B_r[x]$ is a nbd of y. But $y \in X - B_r[x]$ is arbitrary. Therefore $X - B_r[x]$ is a nbd of each of its points. Hence $X - B_r[x]$ is open set. So, $B_r[x]$ is a closed set.

1.5.5 Theorem: Let (X, d) be a metric space. Then

(a) Arbitrary intersection of closed sets in X is closed

(b) Finite union of closed sets in X is closed.