Proof: Let $\{F_\alpha : \alpha \in I\}$ be an arbitrary family of closed sets in X. We claim that $\bigcap_{\alpha \in I} F_\alpha$ is closed. Since F_α is closed for each $\alpha \in I$, $X - F_\alpha$ is open for each $\alpha \in I$. Write $G_\alpha = X - F_\alpha$, $\alpha \in I$. Then

$$\bigcap_{\alpha \in I} F_\alpha = \bigcap_{\alpha \in I}(X - G_\alpha) = X - \bigcup_{\alpha \in I} G_\alpha \quad \text{(by De Morgan's law)}$$

In view of Theorem 1.4.6(a), $\bigcup_{\alpha \in I} G_\alpha$ is open. So, $X - \bigcup_{\alpha \in I} G_\alpha$ is closed and hence $\bigcap_{\alpha \in I} F_\alpha$ is closed.

(b) Let $\{F_i : i = 1, 2, \ldots, n\}$ be a finite family of closed sets. Then, $X - F_i$ is an open set for $i = 1, 2, \ldots, n$. Write $G_i = X - F_i$. Then $\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n}(X - G_i) = X - \bigcap_{i=1}^{n} G_i$ (by De Morgan's Law)

But $\bigcap_{i=1}^{n} G_i$ is open by Theorem 1.4.6(b). So, $X - \bigcap_{i=1}^{n} G_i$ is closed and hence $\bigcup_{i=1}^{n} F_i$ is closed.

Remark: Arbitrary union or even ~~cou~~ countably infinite union of closed sets need not be closed

1.5.6 Example: Consider the family $\{[\frac{1}{n}, 2] : n \in \mathbb{N}\}$ of closed sets in the usual metric space $\mathbb{R}_u$.

Then $\bigcup\{[\frac{1}{n}, 2] : n \in \mathbb{N}\} = (0, 2]$ which is not a closed set.

1.6 INTERIOR POINTS

1.6.1 <u>Definition</u>: Let (X, d) be a metric space and $A \subset X$.

A point $x \in A$ is said to be an interior point of A if A is a nbd of x.

(Equivalently, a point $x \in A$ is said to be an interior point of A if $\exists$ an $r > 0$ such that $B_r(x) \subset A$.)

The interior of A, denoted by Int A or A^0, is the set of all interior points of A. That is,

$$\text{Int } A = A^0 = \{x \in A : B_r(x) \subset A, \text{ for some } r > 0\}$$

1.6.2 Examples :

1. Let $\mathbb{R}_u$ be the usual metric space (Example 1.1.2(1)) and $A \subset \mathbb{R}$. Then :

(a) If $A = (a,b), (a,b], [a,b)$ or $[a,b]$, then

$$A^0 = (a,b)$$

(b) If $A = \mathbb{N}, \mathbb{Z}, \mathbb{Q}$, then

$$A^0 = \phi$$

(c) If $A = \mathbb{R}$ then

$$A^0 = \mathbb{R}$$

2. Let X_d be the discrete metric space (Example 1.1.2(3)) and $A \subset X$, Then

$$A^0 = A.$$

1.6.3 Theorem : Let (X, d) be a metric space and $A \subset X$. Then

(a) A^0 is an open set

(b) A^0 is the largest open subset of A

(c) A is open $\Leftrightarrow A = A^0$

(d) A^0 is the union of all open subsets of A.

Proof: (a) Let $x \in A^0$ be arbitrary. Then, by definition $\exists$ an open ball $B_r(x) \subset A$. But $B_r(x)$ being an open set, each point of $B_r(x)$ is the centre of some open ball contained in $B_r(x)$. Therefore each point of $B_r(x)$ is the interior point of A, i.e. $B_r(x) \subset A^0$. Thus x is the centre of an open ball contained in A^0. But x being arbitrary, it is true for each $x \in A^0$. Hence A^0 is an open set.

(b) Let $G \subset A$ be an open set and $x \in G$ be arbitrary. Then $\exists$ an open ball $S_r(x) \subset G \subset A$. By definition, $x \in A^0$ and as such $G \subset A^0$. Thus given any open set $G \subset A$, we have $G \subset A^0 \subset A$ and A^0 is open by (a). Hence A^0 is the largest open ~~set~~ subset of A.

The assertions (c) and (d) follows obviously from (b).

1.6.A. Theorem

Let (X, d) be a metric space and $A, B \subset X$. Then

(a) $A \subset B \Rightarrow A^0 \subset B^0$

(b) $(A \cap B)^0 = A^0 \cap B^0$

(c) $(A \cup B)^0 \supset A^0 \cup B^0$

Proof: (a) Let $x \in A^0$. Then

$$\exists \text{ an open ball } B_r(x) \subset A$$
$$\Rightarrow B_r(x) \subset B \quad (\because A \subset B)$$
$$\Rightarrow x \in B^0 \quad \Rightarrow A^0 \subset B^0$$

(b) Let $x \in (A \cap B)^{\circ}$. Then $\exists$ an open ball $B_r(x) \subset A \cap B$.

This gives $B_r(x) \subset A$ and $B_r(x) \subset B$

$\Rightarrow$ $x \in A^{\circ}$ and $x \in B^{\circ}$

$\Rightarrow$ $x \in A^{\circ} \cap B^{\circ}$

Hence $(A \cap B)^{\circ} \subset A^{\circ} \cap B^{\circ}$

To establish the reverse inclusion, let $y \in A^{\circ} \cap B^{\circ}$

Then, $y \in A^{\circ}$ and $y \in B^{\circ}$. So, $\exists$

open balls $B_{r_1}(y) \subset A$ and $B_{r_2}(y) \subset B$

$\Rightarrow$ $B_r(y) \subset A \cap B$, $r = \min\{r_1, r_2\}$

$\Rightarrow$ $y \in (A \cap B)^{\circ}$

Consequently $A^{\circ} \cap B^{\circ} \subset (A \cap B)^{\circ}$

So, $(A \cap B)^{\circ} = A^{\circ} \cap B^{\circ}$

(c) Let $x \in A^{\circ} \cup B^{\circ}$. Then $B_r(x) \subset A$ or $B_r(x) \subset B$

for some $r > 0$ and therefore $B_r(x) \subset A \cup B$.

This shows that $x \in (A \cup B)^{\circ}$

Hence $A^{\circ} \cup B^{\circ} \subset (A \cup B)^{\circ}$

Remark: The inclusion relation in 1.6.4 (c) is proper

1.6.5 Example: In the usual metric space $\mathbb{R}_u$, consider the set $A = [0,1]$ and $B = [1,2]$, $A \cup B = [0,2]$

Note that $A^{\circ} = (0,1)$, $B^{\circ} = (1,2)$ and

$(A \cup B)^{\circ} = (0,2)$. So, $A^{\circ} \cup B^{\circ} = (0,1) \cup (1,2)$

This shows that $A^{\circ} \cup B^{\circ} \subsetneq (A \cup B)^{\circ}$