Proof: (a) Let x be a limit point of $\bar{A}$. Then, for a given $\varepsilon>0$, $\exists\, y \in \bar{A}$ such that $d(x,y) < \frac{\varepsilon}{2}$. Further, since $y \in \bar{A}$, i.e., either $y \in A$ or y is a limit point of A, $\exists\, z \in A$ such that $d(y,z) < \frac{\varepsilon}{2}$.

Now $d(x,z) \le d(x,y) + d(y,z) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

This shows that x is a limit point of A and hence $x \in \bar{A}$. This verifies that $\bar{A}$ is a closed set.

(b) It follows from the fact that $\bar{A} = A \cup A'$ and that A is closed if and only if $A' \subset A$

(c) Let B be any closed subset of X with $A \subset B$ and let $x \in \bar{A}$. If $x \in A$, then $x \in B$. In case $x \notin A$, x is a limit point of A. Then, for a given $\varepsilon>0$, $\exists\, y \in A$ such that $d(x,y) < \varepsilon$. But $y \in B$ since $A \subset B$ and therefore x is a limit point of B. The set B being closed, $x \in B$ and as such $\bar{A} \subset B$. Thus, given any closed set $B \supset A$, we have $B \supset \bar{A} \supset A$ and $\bar{A}$ is closed by (a).

Hence $\bar{A}$ is the smallest closed subset of X containing A.

(d) Let $M = \cap B$ where $B \subset X$, B is closed and $B \supset A$

Then by Theorem 1.5.5(a), M is closed. Clearly, M is the smallest closed subset of X containing A. Therefore, by (c), $\bar{A} = M$ which completes the proof.

1.8.3 Theorem

Let (X, d) be a metric space and $A, B \subset X$. Then

(a) $\overline{\phi} = \phi$

(b) $\overline{X} = X$

(c) $\overline{(\overline{A})} = \overline{A}$

(d) $A \subset B \Rightarrow \overline{A} \subset \overline{B}$

(e) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(f) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$

(g) $A' = (\overline{A})'$

Proof: Exercise

1.8.4 Example

In the usual metric space $\mathbb{R}_u$, consider the sets $A = (0,1)$ and $B = (1,2)$. Then,

$A \cap B = \phi$. Note that $\overline{A} = [0,1]$, $\overline{B} = [1,2]$

$$\overline{A} \cap \overline{B} = [0,1] \cap [1,2] = \{1\}$$

$$\overline{A \cap B} = \phi$$

This shows that $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$

1.9 BOUNDARY OF A SET

1.9.1 Definition

Let (X, d) be a metric space and $A \subset X$. A point $x \in X$ is said to be a boundary point of A if x is neither an interior point of A nor of $X - A$. (In other words, $x \in X$ is said to be a boundary point of A if every open ball centred on x intersects both A and $X - A$.)

The set of all boundary points of A, denoted by ∂A, is called the boundary of A. It is also denoted by $b_r(A)$ or $Fr(A)$.

1.9.2 Examples

1. Let $\mathbb{R}_u$ be the usual metric space and $A \subset \mathbb{R}$. Then

(i) If $A = [a,b], [a,b), (a,b]$, or (a,b), then $\partial A = \{a,b\}$

(ii) If $A = \mathbb{N}$ (resp. $\mathbb{Z}$), then $\partial A = \mathbb{N}$ (resp $\mathbb{Z}$)

(iii) If $A = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$, then $\partial A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots\}$

(iv) If $A = \mathbb{Q}$, then $\partial A = \mathbb{R}$

(v) If A is the set of all irrational numbers, then $\partial A = \mathbb{R}$

2. Let X_d be the discrete metric space and $A \subset X$. Then $\partial A = \phi$.

1.9.3 Theorem

Let (X,d) be a metric space and $A \subset X$. Then

(a) $\partial A = \partial(X-A)$

(b) $\partial A = \bar{A} \cap \overline{(X-A)}$

(c) $\partial A = \bar{A} - A^0 = \overline{X-A} - (X-A)^0$

(d) $X - \partial A = A^0 \cup (X-A)^0$

(e) $\bar{A} = A^0 \cup \partial A = A \cup \partial A$

(f) $A^0 \cap \partial A = \phi$

(g) $A^0 = A - \partial A$

(h) A is closed $\Leftrightarrow$ $\partial A \subset A$

(i) A is open $\Leftrightarrow$ $A \cap \partial A = \phi$

Math (664DC) CC13(SB)

Proof: (a) Let $x \in \partial A$. Then, by definition, x is neither an interior point of A nor of $X-A$. Equivalently, x is neither interior point of $X-A$ nor of $X-(X-A)$ $(=A)$. This means that

$$x \in \partial(X-A)$$

Hence

$$\partial A \subset \partial(X-A)$$

The reverse inclusion follows by replacing A with $X-A$ in the above.

(b) Let $x \in \partial A$. Then, x is neither an interior point of A nor of $X-A$. Therefore, $B_r(x) \cap (X-A) = \phi$ and $x \in \overline{(X-A)}$ as well

$B_r(x) \cap A \neq \phi$, for every $r>0$. Hence $\partial A \subset \bar{A} \cap \overline{(X-A)}$.

as $x \in \bar{A}$. This proves that

On the other hand if $x \in \bar{A} \cap \overline{(X-A)}$, then $x \in \bar{A}$ and $x \in \overline{X-A}$ and as such $B_r(x) \cap A \neq \phi$ and $B_r(x) \cap (X-A) \neq \phi$, for every $r>0$. Thus x is neither an interior point of A nor of $X-A$. Therefore, $x \in \partial A$. Hence $\bar{A} \cap \overline{X-A} \subset \partial A$. This verifies (b)

(c) It follows by using (b) and the fact that

$$\overline{X-A} = X - A^{0} \text{ (prove it)}$$

(d)
$$\begin{aligned}
X - \partial A &= X - (\bar{A} \cap \overline{X-A}) \quad (\text{by (b)}) \\
&= (X-\bar{A}) \cup (X - \overline{X-A}) \quad (\text{by De Morgan's Law}) \\
&= (X-A)^{0} \cup (X - (X-A^{0})) \quad (\text{As } \overline{X-A} = (X-A^{0})) \\
&= (X-A)^{0} \cup A^{0}
\end{aligned}$$