(e) We have $A \cup \partial A = A \cup (\bar{A} \cap \overline{X-A})$

$= (A \cup \bar{A}) \cap (A \cup \overline{X-A})$ (by distributive law)

$= \bar{A} \cap X = \bar{A}$

Also $A^{o} \cup \partial A = A^{o} \cup (\bar{A} \cap \overline{X-A})$

$= (A^{o} \cup \bar{A}) \cap (A^{o} \cap \overline{X-A})$ (by distributive law)

$= \bar{A} \cap X = \bar{A}$

(f) Straightforward proof

(g) We know that for any two subsets A and B of X

$A - B = A \cap (X-B)$ Taking $B = \partial A$, we get

$A - \partial A = A \cap (X - \partial A)$

$= A \cap (A^{o} \cup (X-A)^{o})$ (by (d))

$= (A \cap A^{o}) \cup (A \cap (X-A)^{o})$ (by distributive law)

$= A^{o} \cup \phi = A^{o}$

(h) Since in view of (e) above and Theorem 1.8.2(b)

A is closed if and only if $A = A \cup \partial A$, the result follows.

(i) We have

$A$ is open $\Leftrightarrow X - A$ is closed

$\Leftrightarrow \partial(X-A) \subset X - A$ (by (h))

$\Leftrightarrow \partial A \subset X - A$ (by (a))

$\Leftrightarrow A \cap \partial A = \phi$

## 1.10 DISTANCE BETWEEN SETS AND DIAMETER OF A SET

1.10.1 Definition : Let $(X, d)$ be a metric space and let

A, B ⊂ X be non-empty sets.

(a) The distance of a point $x \in X$ from the set A, denoted by $d(x, A)$, is given by

$$d(x, A) = \inf \{ d(x,y) : y \in A \}$$

(b) The distance between the sets A and B, denoted by $d(A, B)$, is given by

$$d(A, B) = \inf \{ d(x,y) : x \in A, y \in B \}$$

(c) The diameter of A, denoted by $d(A)$, is given by

$$d(A) = \sup \{ d(x,y) : x, y \in A \}$$

Note: One should not confuse as we use the symbol d for the metric (distance between two points), distance of a point from a set A, distance between the sets A and B and diameter of a set A.

1.10.2 <u>Theorem</u>

Let $(X, d)$ be a metric space and $A, B \subset X$. Then

(a) $x \in A, y \in B \Rightarrow d(A, B) \leq d(x, y)$

(b) $x \in \bar{A} \Leftrightarrow d(\{x\}, A) = 0$

(c) $d(\bar{A}, \bar{B}) = d(A, B)$

(d) $d(A) = 0 \Leftrightarrow$ A contains atmost one point

(e) $A \subset B \Rightarrow d(A) \leq d(B)$

(f) $d(\bar{A}) = d(A)$

(g) $A \cap B \neq \phi \Rightarrow d(A \cup B) \leq d(A) + d(B)$

(h) $x \in A, y \in B \Rightarrow d(x, y) \leq d(A \cup B)$

(i) $d(A \cup B) \leq d(A) + d(A, B) + d(B)$

Proof: Exercise

1.10.3 Definition : Let $(X, d)$ be a metric space and $A \subset X$. The set $A$ is said to be bounded if $d(A) \leq k < \infty$ for some real $k$. In other words, $A$ is bounded if its diameter is finite, otherwise it is to unbounded. In particular, the metric space $(X, d)$ is bounded if $X$ is bounded

1.10.4 Examples

1. Let $\mathbb{R}_u$ be the usual metric space and $A \subset \mathbb{R}$.

(a) If $A = [a, b]$, $(a, b)$, $[a, b)$ or $(a, b]$, then $A$ is bounded and $d(A) \leq b-a$

(b) If $A = [a, \infty)$ or $[-\infty, a]$, $a \in \mathbb{R}$, then $A$ is not bounded.

2. Let $(X, d)$ be a metric space. If $A = B_r(x)$ or $B_r[x]$, where $x \in X$ and $r > 0$, then $A$ is bounded and $d(A) \leq 2r$

3. Every set in a discrete metric space is bounded (verify it)

1.10.5. Theorem

Let $(X, d)$ be a metric space and $A \subset X$. Then the following statements are equivalent :

(a) $A$ is bounded

(b) $\exists\ k > 0$ such that $d(x, y) \leq k$, $\forall\ x, y \in A$

(c) $\exists\ x_0 \in X$ and $r_0 > 0$ such that $A \subset B_{r_0}[x_0]$

(d) For every $x \in X$, $\exists$ an $r > 0$ such that $A \subset B_r[x]$

Proof: Exercise

1.11 SUBSPACE OF A METRIC SPACE

1.11.1 Definition: Let $(X, d)$ be a metric space and $Y \subset X$.

The mapping $d_Y : Y \times Y \to \mathbb{R}$ given by

$$d_Y(x,y) = d(x,y), \ \forall x, y \in Y$$ is

metric on $Y$. The metric $d_Y$ is called the relative metric induced (or simply the metric induced) on $Y$ by $d$.

The space $(Y, d_Y)$ is called the metric subspace of the metric space $(X, d)$.

The above method of forming subspace of a given metric space enables us to construct several examples of metric spaces.

1.11.2 Examples

1. The intervals $[0,1]$, $(0,1)$, $(0,1]$, $[0,1)$, the set $\mathbb{Q}$ etc. are the subspaces of the metric space $\mathbb{R}_u$. The metric space $\mathbb{R}_u$ itself is a subspace of the metric space $\mathbb{R}_u$

2. The space $\mathbb{R}$ is a subspace of $\mathbb{R}^2$

3. The set $\mathcal{P}[a,b]$ of all polynomials defined on $[a,b]$ is a subspace of the metric space $C[a,b]$ with uniform metric $d_\infty$ (Example 2.1.2(15))

Remark: If $Y$ is a subspace of a metric space $X$,