Notes on my portion of CC7

SB (Subhasundar Bandyopadhyay)
Deptt. of Mathematics, GGGDC

Books that are to be followed :

1. Advanced Differential Calculus of several variables - S. K. Mukherjee
2. Mathematical Analysis - S. C. Malik & S. Arora
3. Horst R. Beyer, Calculus and Analysis
4. Multivariable Analysis - S. Shirali & H. L. Vasudeva

The portion of my syllabus in CC 7 is

Unit 2 : Multivariate Calculus - I :

- Concept of neighbourhood of a point in $\mathbb{R}^n$ $(n>1)$, interior point, limit point, open set and closed set in $\mathbb{R}^n$ $(n>1)$
- Functions from $\mathbb{R}^n$ $(n>1)$ to $\mathbb{R}^m$ $(m\geq 1)$, limit and continuity of functions of two or more variables. Partial derivatives, total derivative and differentiability, sufficient condition for differentiability. Chain rule for one and two independent parameters, directional derivatives, the gradient, maximal and normal property of the gradient, tangent planes. Extrema of functions of two variables, method of Lagrange multipliers. Constrained optimization problems.

1. $\mathbb{R}^n$ and the distance in it.

Let $\mathbb{R}^n = \{x=(x_1,x_2,\ldots,x_n) : x_1,x_2,\ldots,x_n \in \mathbb{R}\}$ be the real Euclidean space with the standard inner product defined by

$$x\cdot y = \sum_{i=1}^{n} x_i y_i \quad , \; x=(x_1,x_2,\ldots,x_n),\; y=(y_1,y_2,\ldots,y_n)\in \mathbb{R}^n$$

Norm (or length) of x is defined by (denoted by $\|x\|$)

$$\|x\| = \sqrt{x\cdot x} = \sqrt{\sum_{i=1}^{n} x_i^2} \quad \text{where } x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$$

$d : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ defined by $d(x,y) = \|x-y\| = \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}$,

$x=(x_1,x_2,\ldots,x_n)$ and $y=(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n$ is a metric or distance as (i) $d(x,y)\geq 0$,

(ii) $d(x,y) = 0 \Leftrightarrow x = y$

(iii) $d(x,y) = d(y,x)$

(iv) $d(x,z) \leq d(x,y) + d(y,z)$ for $x,y,z \in \mathbb{R}^n$

1.1. Open and closed sets in $\mathbb{R}^n$

Definition 1.1.1 For $\delta > 0$, the set $B(a,\delta) = \{x \in \mathbb{R}^n : d(a,x) < \delta\}$ is called ~~the~~ an open ball with centre $a \in \mathbb{R}^n$ of radius δ or the δ-neighbourhood of the point $a \in \mathbb{R}^n$.

In particular, if $(a,b) \in \mathbb{R}^2$ and $\delta > 0$, the set $\{(x,y) \in \mathbb{R}^2 : (x-a)^2 + (y-b)^2 < \delta^2$ is called an open disc of radius δ with centre at (a,b) and is denoted by $N((a,b),\delta)$.

The set $N'((a,b),\delta) = N((a,b),\delta) - \{(a,b)\}$ is called a deleted δ-neighbourhood of (a,b).

Definition 1.1.2 Let $E \subset \mathbb{R}^n$

(i) A point $x \in E$ is said to an interior point of E if there is ~~some~~ a $B(x,\delta)$ for some $\delta > 0$ such that $B(x,\delta) \subset E$

(ii) A point $x \in \mathbb{R}^n$ is said to be an exterior point if it is an interior point of the complement of E in $\mathbb{R}^n$.

(iii) A point $x \in \mathbb{R}^n$ is said to be a boundary point of E if it neither an interior point of E nor an exterior point of E

Definition 1.1.3 A point $a \in \mathbb{R}^n$ is said to be a limit point of the set $E \subset \mathbb{R}^n$ if $B(a,\delta) \cap E$ is an infinite set for each $\delta > 0$.

Definition 1.1.4 The union of a set E and all its limit points in $\mathbb{R}^n$ is called the closure of E and denoted by $\overline{E}$

Definition 1.1.5

A set $G \subset \mathbb{R}^n$ is said to be open in $\mathbb{R}^n$ if for every point $x \in G$, there is an open ball $B(x, \delta)$ such that $B(x, \delta) \subset G$.

Examples: (1) $\mathbb{R}^n$ is an open set in $\mathbb{R}^n$

(2) The empty set ϕ contains no point and may be regarded as an open set.

(3) An open ball $B(a, r)$ is an open set in $\mathbb{R}^n$. Because, if $x \in B(a, r)$ then $d(a, x) < r$, then for $0 < \delta < r - d(a, x)$, we have $B(x, \delta) \subset B(a, r)$.

(4) A set $G = \{x \in \mathbb{R}^n : d(a, x) > r\}$ i.e., the set of points in $\mathbb{R}^n$ whose distance from a fixed point $a \in \mathbb{R}^n$ is larger than r, is open.

Definition 1.1.6 The set $F \subset \mathbb{R}^n$ is said to be closed if its complement $G = \mathbb{R}^n \setminus F$ is open in $\mathbb{R}^n$.

Example The set $\overline{B}(a, r) = \{x \in \mathbb{R}^n : d(a, x) \leq r\}, r > 0$, is closed by definition 1.1.6 and example (4).

1.2 Some results regarding open sets and closed sets in $\mathbb{R}^2$

Result 1.2.1 Intersection of any two open discs is an open set

Proof: Let $D_1 = \{q \in \mathbb{R}^2 : d(p_1, q) < \delta_1\}$

and $D_2 = \{q \in \mathbb{R}^2 : d(p_2, q) < \delta_2\}$ where $\delta_1, \delta_2 > 0$, are given open discs.

If $D_1 \cap D_2 = \phi$, then it is an open set.

Let $p_0 \in D_1 \cap D_2$. Then $d(p_1, p_0) < \delta$ and $d(p_2, p_0) < \delta_2$

we take $r = \min\{\delta_1 - d(p_1, p_0), \delta_2 - d(p_2, p_0)\}$

and let $D = \{q \in \mathbb{R}^2 : d(p_0, q) < \frac{1}{2} r\}$

$\Rightarrow p_0 \in D \subset D_1 \cap D_2 \Rightarrow p_0$ is an interior point of $D_1 \cap D_2$.

So, every point of $D_1 \cap D_2$ is its interior point. Consequently, $D_1 \cap D_2$ is open.

Result 1.2.2 The union of any number of open subsets of $\mathbb{R}^2$ is open.

Proof: Let $\{G_\alpha : \alpha \in I\}$ be an arbitrary family of open subsets of $\mathbb{R}^2$

Let $H = \bigcup_{\alpha \in I} G_\alpha$. To show that H is open

Let $p \in H$. So, $\exists$ $\beta \in I$ such that $p \in G_\beta$.

As G_β is open, p is an interior point of G_β

So, $\exists$ an open disc D_p containing p such that

$D_p \subset G_\beta \Rightarrow D_p \subset H$. So, p is an interior point of H. So, H is an open set in $\mathbb{R}^2$

Result 1.2.3 The intersection $\bigcap_{i=1}^{n} G_i$ of a finite number open sets G_i, $i=1,2,\dots,n$ in $\mathbb{R}^2$ is an open set in $\mathbb{R}^2$.

Proof: If $\bigcap_{i=1}^{n} G_i = \phi$, then it is an open set.

Let $p \in \bigcap_{i=1}^{n} G_i$. Then $p \in G_i$, $i = 1, 2, \dots n$

Let $\delta_1, \delta_2, \dots, \delta_n$ be positive numbers such that