In 2-space ($\mathbb{R}^2$) the gradient vector is often written as $\nabla f(x,y) = \frac{\partial f(x,y)}{\partial x}\hat{i} + \frac{\partial f(x,y)}{\partial y}\hat{j}$. Here $\hat{i} = (1,0)$, $\hat{j} = (0,1)$

In 3-space ($\mathbb{R}^3$), the corresponding formula is

$$\nabla f(x,y,z) = \frac{\partial f(x,y,z)}{\partial x}\hat{i} + \frac{\partial f(x,y,z)}{\partial y}\hat{j} + \frac{\partial f(x,y,z)}{\partial z}\hat{k}$$

Here $\hat{i} = (1,0,0)$, $\hat{j} = (0,1,0)$ and $\hat{k} = (0,0,1)$

<u>1.10</u> A sufficient condition for differentiability

If $f : S \to \mathbb{R}$, $S \subseteq \mathbb{R}^n$, is differentiable at a, then all the partial derivatives $D_1 f(a), D_2 f(a), \ldots, D_n f(a)$ exist. However, the existence of all these partial derivatives does not necessarily imply that f is differentiable at a.

A counterexample is provided by the function

$$f(x,y) = \frac{xy^2}{x^2+y^4} \text{ if } x \neq 0, \quad f(0,y) = 0$$ in Page-20

of section 1.7. For this function, both partial derivatives $D_1 f(0,0)$ and $D_2 f(0,0)$ exist but f is not continuous at $(0,0)$ and hence f can not be differentiable at $(0,0)$.

Theorem 1.10.1 (A SUFFICIENT CONDITION FOR DIFFERENTIABILITY)
Let $f : S \to \mathbb{R}$ be a function where $S \subseteq \mathbb{R}^n$. Assume that partial derivatives $D_1 f, D_2 f, \ldots, D_n f$ exist in some n-ball $B(a) \subseteq S$ (where $B(a) = \{x \in \mathbb{R}^n : \|x-a\| < r\}$, r is a positive real number) and are continuous at a. Then f is differentiable at a.

(Note: A function $f : S \to \mathbb{R}$, $S \subseteq \mathbb{R}^n$ satisfying the hypothesis of Theorem 1.10.1 is said to be continuously differentiable (at a))

Proof: The only candidate for $T_a(v)$ is $\nabla f \cdot v$. We will show that $f(a+v) - f(a) = \nabla f \cdot v + \|v\| E(a,v)$, where $E(a,v) \to 0$ as $\|v\| \to 0$. This will prove the theorem.

Let $\lambda = \|v\|$. Then $v = \lambda u$, where $\|u\| = 1$. We keep λ small enough so that $a+v$ lies in the Ball $B(a)$ in which the partial derivatives $D_1 f, D_2 f, \dots, D_n f$ exist.

Expressing u in terms of its components we have

$u = u_1 e_1 + \dots + u_n e_n$,

where $e_1, e_2, \dots, e_n$ are the unit co-ordinate vectors. Now we write the difference $f(a+v) - f(a)$ as a telescopic sum

$$f(a+v) - f(a) = f(a+\lambda u) - f(a) = \sum_{k=1}^{n} \{ f(a+\lambda v_k) - f(a+\lambda v_{k-1}) \}, \quad \text{---(1)}$$

where $v_0, v_1, \dots, v_n$ are any vectors in $\mathbb{R}^n$ such that $v_0 = 0 = (\overbrace{0,0,\dots,0}^{n\text{ terms}})$ and $v_n = u$. We choose these vectors so that they satisfy the recurrence relation

$v_k = v_{k-1} + u_k e_k$. That is, we take

$v_0 = 0$, $v_1 = u_1 e_1$, $v_2 = u_1 e_1 + u_2 e_2$, $\dots$, $v_n = u_1 e_1 + u_2 e_2 + \dots + u_n e_n$

Then the kth term of the sum (1) becomes

$$f(a+\lambda v_{k-1} + \lambda u_k e_k) - f(a+\lambda v_{k-1}) = f(b_k + \lambda u_k e_k) - f(b_k)$$

where $b_k = a + \lambda v_{k-1}$. The two points b_k and $b_k + \lambda u_k e_k$ differ only in their kth component. Therefore we can apply the mean value theorem of differential calculus to write

$f(b_k + \lambda u_k e_k) - f(b_k) = \lambda u_k D_k f(c_k)$ (2) where c_k lies on the ~~lines joining~~ line segment joining b_k

to $b_k + \lambda u_k e_k$. Note that $b_k \to a$ and hence $c_k \to a$ as $\lambda \to 0$

Using (2) in (1), we obtain

$$f(a+v) - f(a) = \lambda \sum_{k=1}^{n} D_k f(c_k) u_k$$

But $\nabla f(a) \cdot v = \lambda \nabla f(a) \cdot u = \lambda \sum_{k=1}^{n} D_k f(a) u_k$, so

$$f(a+v) - f(a) - \nabla f(a) \cdot v = \lambda \sum_{k=1}^{n} \left(D_k f(c_k) - D_k f(a)\right) u_k = \|v\| E(a,v),$$

where $E(a,v) = \sum_{k=1}^{n} \left(D_k f(c_k) - D_k f(a)\right) u_k$.

Since $c_k \to a$ as $\|v\| \to 0$ and since each partial derivative $D_k f$ is continuous at a, we see that $E(a,v) \to 0$ as $\|v\| \to 0$. This completes the proof.

When f is a function $f: S \to \mathbb{R}$ where S is an open set in $\mathbb{R}^2$ we can define differentiability at an interior point (a,b) of S as follows: We say f is differentiable at (a,b) if $\Delta f = f(a+h, b+k) - f(a,b) = Ah + Bk + h\phi(h,k) + k\psi(h,k)$ where $(a+h, b+k) \in S$, A and B are independent of h and k and $\phi(h,k) \to 0$ and $\psi(h,k) \to 0$ as $(h,k) \to (0,0)$.

<u>Theorem 1.10.2</u> If f is differentiable at an interior point (a,b) of its domain then (i) f_x and f_y exist at that point and (ii) f is continuous at that point.

Proof: By hypothesis

$$f(a+h, b+k) - f(a,b) = Ah + Bk + h\phi(h,k) + k\psi(h,k) \quad \ldots (i)$$

where $(a+h, b+k)$ is a point of domain of f, A and B are independent of h and k and $\phi(h,k) \to 0$ and $\psi(h,k) \to 0$ as $(h,k) \to (0,0)$

~~If~~ we take $k=0$, $h \neq 0$

Now $\dfrac{f(a+h,b)-f(a,b)}{h} = A + \phi$

As $h \to 0$, $\phi \to 0$. So, $\lim\limits_{h\to 0} \dfrac{f(a+h,b)-f(a,b)}{h}$ exists and equal to

$A \Rightarrow f_x(a,b) = A$

We take $h=0$, $k \neq 0$

then $\dfrac{f(a,b+k)-f(a,b)}{k} = B + \psi$

As $k \to 0$, $\psi \to 0$. So, $\lim\limits_{k\to 0} \dfrac{f(a,b+k)-f(a,b)}{k}$ exists and is equal

to $B \Rightarrow f_y(a,b) = B$. So, (i) is proved

From (i) $\lim\limits_{(h,k)\to(0,0)} f(a+h, b+k) = f(a,b)$.

$\Rightarrow f$ is continuous at (a,b)

Example 1 Let $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$

Now $\lim\limits_{h\to 0} \dfrac{f(0+h,0)-f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{0-0}{h} = 0 \Rightarrow f_x(0,0) = 0$

$\lim\limits_{k\to 0} \dfrac{f(0,0+k)-f(0,0)}{k} = \lim\limits_{k\to 0} \dfrac{0-0}{k} = 0 \Rightarrow f_y(0,0) = 0$

In order to be differentiable at $(0,0)$ we must

have $f(0+h,0+k)-f(0,0) = h f_x(0,0) + k f_y(0,0) + h\phi + k\psi$

where $\phi \to 0$ and $\psi \to 0$ as $(h,k)\to(0,0)$

In particular, if $h = k \neq 0$

$$\frac{h^2}{h\sqrt{2}} = h(\phi+\psi) \Rightarrow \frac{1}{\sqrt{2}} = \phi + \psi$$

As $h \to 0$, $\phi + \psi \to 0$ but the left hand side is $\frac{1}{\sqrt{2}}$

Thus we arrive at an absurd stage.

So, f is not differentiable at $(0,0)$ though