$f_x(0,0)$ and $f_y(0,0)$ exist.

We also give a sufficient condition for differentiability at a point here.

Theorem 1.10.3 Let $f : S \to \mathbb{R}$ where $S \subseteq \mathbb{R}^2$

If (a,b) be a point of domain of definition of f such that

(i) f_y is continuous at (a,b)

(ii) f_x exists at (a,b),

then f is differentiable at (a,b)

Proof: By hypothesis, there exists a neighbourhood $N((a,b),\delta)$ in which f and f_y are defined. We take h, k; $h^2+k^2 \neq 0$, so that $(a+h, b+k), (a+h, b) \in N((a,b),\delta)$

We take

$$f(a+h, b+k) - f(a,b) = \left[f(a+h,b+k) - f(a+h,b)\right] + \left[f(a+h,b) - f(a,b)\right]$$

As $f_x(a,b)$ exists, $\lim\limits_{h\to 0} \dfrac{f(a+h,b) - f(a,b)}{h} = f_x(a,b)$

$$\Rightarrow \frac{f(a+h,b) - f(a,b)}{h} - f_x(a,b) = \epsilon_1 \quad \text{where } \epsilon_1 \to 0 \text{ as } h \to 0$$

$$\Rightarrow f(a+h,b) - f(a,b) = h\left(f_x(a,b) + \epsilon_1\right) \quad \text{where } \epsilon_1 \to 0 \text{ as } h \to 0$$

Let $f(a+h, y) = g(y)$. Then $f(a+h, b+k) - f(a+h,b) = g(b+k) - g(b)$

As f_y exists in $N((a,b),\delta)$, g is derivable in $[b, b+k]$ or $[b+k, b]$.

By Lagrange's mean value theorem, $\exists\ \theta \in (0,1)$ such that

$$g(b+k) - g(b) = k\, g'(b+\theta k)$$

$$\Rightarrow f(a+h, b+k) - f(a+h, b) = k\, f_y(a+h, b+\theta k).$$

Consequently

$$f(a+h, b+k) - f(a,b) = h f_x(a,b) + \epsilon_1 h + k f_y(a+h, b+\theta k)$$

As f_y is continuous at (a,b)

$$\lim_{(h,k)\to(0,0)} f_y(a+h, b+\theta k) = f_y(a,b)$$

So, if $f_y(a+h, b+\theta k) - f_y(a,b) = \epsilon_2(h,k)$, then

$\epsilon_2 \to 0$ as $(h,k) \to (0,0)$

Hence we get

$$f(a+h, b+k) - f(a,b) = h f_x(a,b) + k f_y(a,b) + \epsilon_1 h + \epsilon_2 k$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $(h,k) \to (0,0)$

Hence f is differentiable at (a,b)

Note 1: If f is not differentiable at (a,b), the partial derivatives can not be continuous at (a,b)

Note 2: The condition of continuity of one of the partial derivatives at the point is not necessary for differentiability of the function at that point

For Example, let
$$f(x,y) = \begin{cases} x^2 \sin\frac{1}{x} + y^2 \sin\frac{1}{y}, & xy \neq 0 \\ x^2 \sin\frac{1}{x}, & x \neq 0, y = 0 \\ y^2 \sin\frac{1}{y}, & x = 0, y \neq 0 \\ 0, & x = 0, y = 0 \end{cases}$$

As $\lim_{t\to 0} t \sin\frac{1}{t} = 0$, so $f_x(0,0) = 0 = f_y(0,0)$

If we take $\phi(h,k) = h \sin\frac{1}{h}$, $h \neq 0$ and
$= 0$, $h = 0$

and $\psi(h,k) = k \sin\frac{1}{k}$, $k \neq 0$ and
$= 0$, $k = 0$

Then $\phi \to 0$, $\psi \to 0$ as $(h,k) \to (0,0)$

So, $f(h,k) - f(0,0) = h^2 \sin\frac{1}{h} + k^2 \sin\frac{1}{k}$

$= h\phi + k\psi$

where $\phi \to 0$ and $\psi \to 0$ as $(h,k) \to (0,0)$

So, $f(x,y)$ is differentiable at $(0,0)$

Here $f_x(x,y) = \begin{cases} 2x \sin\frac{1}{x} - \cos\frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

$f_y(x,y) = \begin{cases} 2y \sin\frac{1}{y} - \cos\frac{1}{y} & , y \neq 0 \\ 0 & , y = 0 \end{cases}$

So, Both f_x and f_y are discontinuous at $(0,0)$ as

$\lim_{t\to 0}$ ~~Sin~~ $\cos\frac{1}{t}$ does not exist.

~~Chain Rule~~

Definition: If $f(x,y)$ be a differentiable function,

$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ is the total differential of f. and denoted by df.

Examples 1. Find $df(1,2)$ where $f(x,y) = x^2 + xy + y^2 - 4\log x - 10\log y$

Solution: $\frac{\partial f}{\partial x} = 2x + y - \frac{4}{x}$, $\frac{\partial f}{\partial y} = x + 2y - \frac{10}{y}$

At $(1,2)$ $f_x = 0$, $f_y = 0$. So, $df(1,2) = 0$

2. Show that expression $(3x+y)dx + (x+3y)dy$ is a total differential of some function

Solution: Let $du(x,y) = (3x+y)dx + (x+3y)dy$

As $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$, so $\frac{\partial u}{\partial x} = 3x+y$, $\frac{\partial u}{\partial y} = x+3y$

From the first, $u = \frac{3x^2}{2} + xy + \phi(y) \Rightarrow \frac{\partial u}{\partial y} = x + \phi'(y) = x+3y$

So, $\phi(y) = \frac{3y^2}{2} + c$, c is a real constant

Hence $u(x,y) = \frac{3x^2}{2} + xy + \frac{3y^2}{2} + c$

Chain rule

Theorem 1.10.4 Let (i) $x = \phi(u,v)$, $y = \psi(u,v)$ be two functions of u, v defined in a domain $S \subseteq \mathbb{R}^2$ and differentiable at point (u,v) of S (ii) $z = f(x,y)$ be defined on $S_1 \subseteq \mathbb{R}^2$ and differentiable at (x,y) of S_1 (iii) S_1 be the image set of S, then z, defined as a function of u, v, is differentiable at the corresponding point (u,v)

Proof: As $z = f(x,y)$ is differentiable,

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y) = f_x \cdot \Delta x + f_y \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y \quad \text{---(1)}$$

where $\epsilon_1 \to 0$, $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

As $x = \phi(u,v)$, $y = \psi(u,v)$ are differentiable

$$\Delta x = \phi(u+\Delta u, v+\Delta v) - \phi(u,v) = \phi_u \cdot \Delta u + \phi_v \Delta v + \epsilon_3 \Delta u + \epsilon_4 \Delta v \cdots (2)$$

where $\epsilon_3 \to 0$, $\epsilon_4 \to 0$ as $(\Delta u, \Delta v) \to (0,0)$

$$\Delta y = \psi(u+\Delta u, v+\Delta v) - \psi(u,v) = \psi_u \Delta u + \psi_v \Delta v + \epsilon_5 \Delta u + \epsilon_6 \Delta v \quad \text{---(3)}$$

where $\epsilon_5 \to 0$, $\epsilon_6 \to 0$ as $(\Delta u, \Delta v) \to (0,0)$

Consequently, by (1), (2) and (3)

$$\Delta z = (f_x + \epsilon_1)(\phi_u \Delta u + \phi_v \Delta v + \epsilon_3 \Delta u + \epsilon_4 \Delta v) + (f_y + \epsilon_2)(\psi_u \Delta u + \psi_v \Delta v + \epsilon_5 \Delta u + \epsilon_6 \Delta v)$$

$$= (f_x\phi_u + f_y\psi_u)\Delta u + (f_x\phi_v + f_y\psi_v)\Delta v + \{f_x\epsilon_3 + \epsilon_1\phi_u + \epsilon_1\epsilon_3 + f_y\epsilon_5 + \epsilon_2\psi_u + \epsilon_2\epsilon_5\}\Delta u + \{f_x\epsilon_4 + \epsilon_1\phi_v + \epsilon_1\epsilon_4 + f_y\epsilon_6 + \epsilon_2\psi_v + \epsilon_2\epsilon_6\}\Delta v$$

$$= (f_x\phi_u + f_y\psi_u)\Delta u + (f_x\phi_v + f_y\psi_v)\Delta v + \epsilon_7 \Delta u + \epsilon_8 \Delta v$$

where $\epsilon_7 = f_x\epsilon_3 + \epsilon_1\phi_u + \epsilon_1\epsilon_3 + f_y\epsilon_5 + \epsilon_2\psi_u + \epsilon_2\epsilon_5$

and $\epsilon_8 = f_x\epsilon_4 + \epsilon_1\phi_v + \epsilon_1\epsilon_4 + f_y\epsilon_6 + \epsilon_2\psi_v + \epsilon_2\epsilon_6$