where $\alpha(t)$ and $\beta(t)$ are the angles made by the vector $T(t)$ and the positive x-axis and y-axis; the directional derivative of f along C becomes

$$\nabla f(r(t)) \cdot T(t) = D_1 f(r(t)) \cos\alpha(t) + D_2 f(r(t)) \cos\beta(t).$$

This formula is often written more briefly as

$$\nabla f \cdot T = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\cos\beta.$$

Some author write $\frac{df}{ds}$ for the directional derivative $\nabla f \cdot T$. Since the directional derivative along C is defined in terms of T, its value depends on the parametric representation chosen for C. A change of the representation could reverse the direction of T; this, in turn, would reverse the sign of the directional derivative.

Example 2 Find the directional derivative of the scalar field $f(x,y) = x^2 - 3xy$ along the parabola $y = x^2 - x + 2$ at the point $(1,2)$.

Solution: At an arbitrary point (x,y) the gradient vector is $\nabla f(x,y) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} = (2x-3y)\hat{i} - 3x\hat{j}$

At the point $(1,2)$ we have $\nabla f(1,2) = -4\hat{i} - 3\hat{j}$. The parabola can be represented parametrically by the vector equation $r(t) = t\hat{i} + (t^2 - t + 2)\hat{j}$. Therefore,

$r(1) = \hat{i} + 2\hat{j}$, $r'(t) = \hat{i} + (2t-1)\hat{j}$ and $r'(1) = \hat{i} + \hat{j}$. For this representation of C the unit tangent vector T(1) is $(\hat{i}+\hat{j})/\sqrt{2}$ and the required directional derivative is

$\nabla f(1,2) \cdot T(1) = -7/\sqrt{2}$.

Example 3. Let f be a nonconstant scalar field, differentiable everywhere in the plane, and let c be a constant. Assume the Cartesian equation $f(x,y) = c$ describes a curve C having a tangent at each of its points. Prove that f has the following properties at each point of C:

(a) The gradient vector ∇f is normal to C.

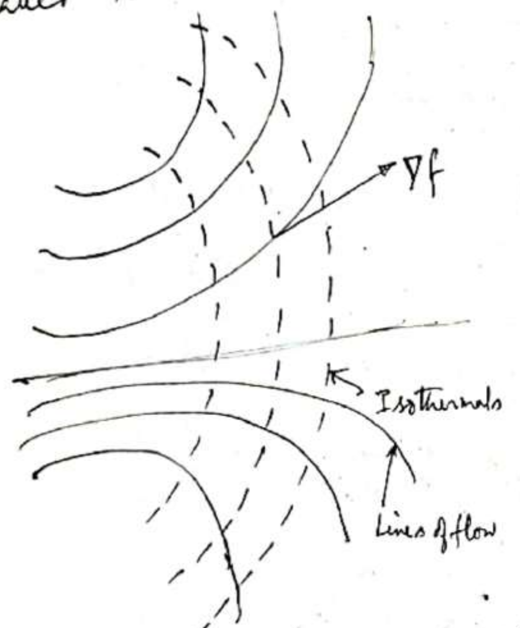
(b) The directional derivative of f is zero along C.

(c) The directional derivative of f has its largest value in a direction normal to C.

Solution: If T is unit tangent vector ~~of~~ to C, the directional derivative of f along C is the dot product $\nabla f \cdot T$. This product is zero if ∇f is perpendicular to T, and it has its largest value if ∇f is parallel to T. Therefore both statements (b) and (c) are consequences of (a). To prove (a), consider any plane curve Γ with a vector equation of the form $r(t) = X(t)\hat{i} + Y(t)\hat{j}$ and introduce the function $g(t) = f(r(t))$. By the chain rule we have $g'(t) = \nabla f(r(t)) \cdot r'(t)$, when $\Gamma = C$, the function g has the constant value c so $g'(t) = 0$ if $r(t) \in C$. Since $g' = \nabla f \cdot r'$, this shows that ∇f is perpendicular to r' on C; hence ∇f is normal to C.

1.12 Application to geometry. Level sets. Tangent planes

The chain rule can be used to deduce geometric properties of the gradient vector. Let f be a scalar field defined on a set S in $\mathbb{R}^n$, i.e $f : S \to \mathbb{R}$, where $S \subseteq \mathbb{R}^n$ and consider those points x in S for which $f(x)$ has a constant value, say $f(x) = c$. Denote this set by $L(c)$, so that $L(c) = \{x : x \in S \text{ and } f(x) = c\}$. The set $L(c)$ is called a level set of f; In $\mathbb{R}^2$, $L(c)$ is called a level curve; in $\mathbb{R}^3$ it is called a level surface.



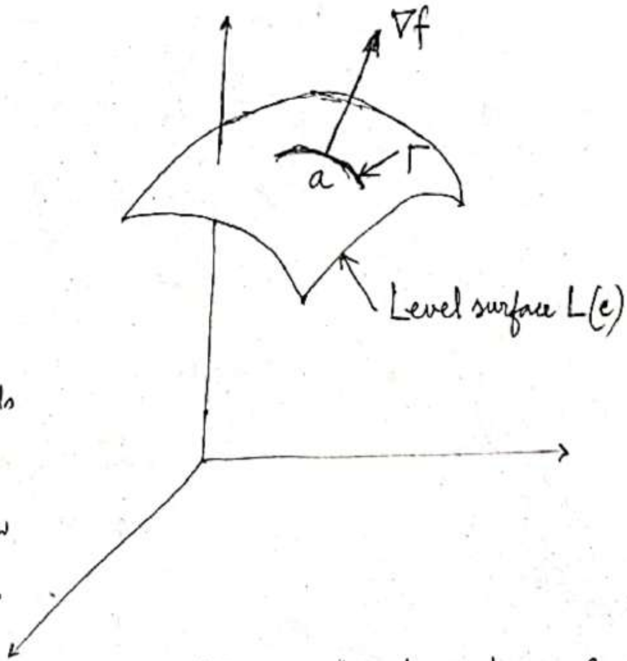


Figure 1. The dotted curves are isothermals: $f(x,y) = c$. The gradient vector ∇f points in the direction of flow.

Figure 2. The gradient vector ∇f is normal to each curve Γ on the level surface $f(x,y,z) = c$.

Families of level sets occur in many physical applications. For example, if $f(x,y)$ represents temperature at (x,y), the level curves of f (curves of constant temperature) are called isothermals. The flow of heat takes place in the direction of most rapid change in temperature. As was shown in Example 3 of the foregoing section, this direction is normal to the isothermals. Hence, in a thin flat sheet the flow of the heat is along a family of curves orthogonal to the isothermals. These are called the lines of flow; they are orthogonal trajectories of the isothermals. Examples are shown in figure 1.

Now consider a scalar field f differentiable on an open set S in $\mathbb{R}^3$, and examine one of its level surfaces, $L(c)$. Let a be a point on this surface, and consider a curve Γ which lies on the surface and passes through a, as suggested by figure 2. We shall prove that the gradient vector $\nabla f(a)$ is normal to this curve at a. That is, we shall prove that $\nabla f(a)$ is perpendicular to the tangent vector of Γ at a.

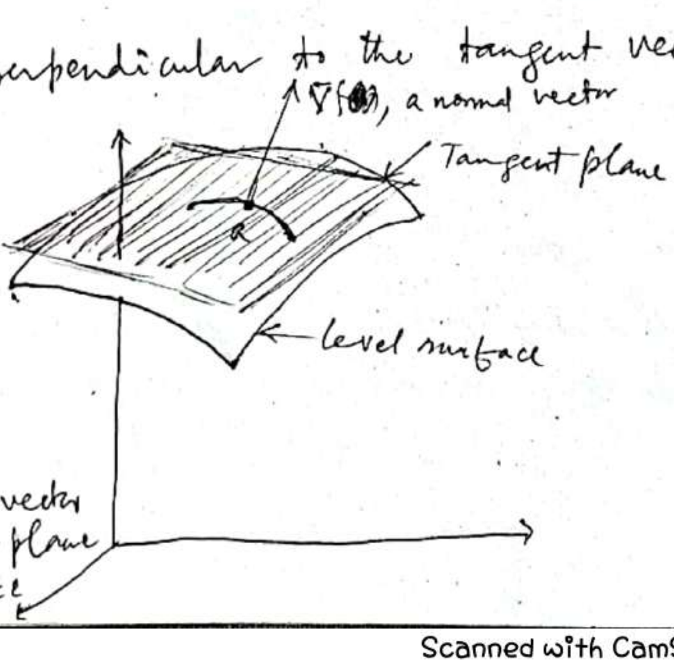


Figure 3 The gradient vector ∇f is normal to the tangent plane of a level surface $f(x,y,z)=c$