For this purpose, we assume that Γ is described parametrically by a differentiable vector-valued function r defined on some interval J of $\mathbb{R}$. Since Γ lies on the level surface $L(c)$, the function r satisfies the equation

$$f(r(t)) = c \quad \text{for all } t \text{ in } J.$$

If $g(r) = f(r(t))$ for t in J, the chain rule states that $g'(t) = \nabla f(r(t)) \cdot r'(t)$.

Since g is constant on J, we have $g'(t) = 0$ on J.

In particular, choosing t_1 so that $g(t_1) = a$

we find that $\nabla f(a) \cdot r'(t_1) = 0$

In other words, the gradient of f at a is perpendicular to the tangent vector $r'(t_1)$, as asserted.

Now, we take a family of curves on the level surface $L(c)$, all passing through the point a. According to the foregoing discussion, the tangent vectors of all these curves are perpendicular to the gradient vector $\nabla f(a)$. If $\nabla f(a)$ is not the zero vector, these tangent vectors determine a plane, and the gradient $\nabla f(a)$ is normal to this plane (see figure 3). This particular plane is called the tangent plane of the surface $L(c)$ at a.

We know that a plane through a with normal vector N consists of all points x in $\mathbb{R}^3$

satisfying $N \cdot (x-a) = 0$. Therefore the tangent plane to the level surface $L(c)$ at a consists of all x in $\mathbb{R}^3$ satisfying $\nabla f(a) \cdot (x-a) = 0$

To obtain a Cartesian equation for this plane we express x, a and $\nabla f(a)$ in terms of their components.

Writing $x = (x, y, z)$, $a = (x_1, y_1, z_1)$, and

$$\nabla f(a) = D_1 f(a)\hat{i} + D_2 f(a)\hat{j} + D_3 f(a)\hat{k},$$

we obtain the Cartesian equation

$$D_1 f(a)(x-x_1) + D_2 f(a)(y-y_1) + D_3 f(a)(z-z_1) = 0$$

A similar discussion applies to scalar fields defined in $\mathbb{R}^2$. In example 3 of the foregoing section we proved that the gradient vector $\nabla f(a)$ at a point a of a level curve is perpendicular to the tangent vector of the curve at a. Therefore the tangent line of the level curve $L(c)$ at a point $a = (x_1, y_1)$ has the Cartesian equation

$$D_1 f(a)(x-x_1) + D_2 f(a)(y-y_1) = 0.$$

1.13 Derivatives of vector fields.

Derivative theory for vector fields is a straightforward extension of that for scalar fields. Let $f: S \to \mathbb{R}^m$ be a vector field defined on a subset S of $\mathbb{R}^n$. If a is an interior point of S and if y is any vector in

$\mathbb{R}^n$ we define the derivative $f'(a;y)$ by the formula

$$f'(a;y) = \lim_{h\to 0} \frac{f(a+hy) - f(a)}{h},$$

whenever the limit exists. The derivative $f'(a;y)$ is a vector in $\mathbb{R}^m$.

Let f_k denotes the kth component of f. We note that the derivative $f'(a;y)$ exists if and only if $f_k'(a;y)$ exists for each $k = 1, 2, \ldots, m$, in which case we have

$$f'(a;y) = \left(f_1'(a;y), f_2'(a;y), \ldots, f_m'(a;y)\right)$$
$$= \sum_{k=1}^{m} f_k'(a;y) e_k,$$

where e_k is the kth unit coordinate vector.

We say that f is differentiable at an interior point a if there is a linear transformation

$$T_a : \mathbb{R}^n \to \mathbb{R}^m$$

such that $f(a+v) = f(a) + T_a(v) + \|v\| E(a,v)$ --- (1)

where $E(a,v) \to 0$ as $v \to 0$. The first order Taylor formula (1) is to hold for all v with $\|v\| < r$ for some $r > 0$. The term $E(a,v)$ is a vector in $\mathbb{R}^m$.

The linear transformation T_a is called the total derivative of f at a.

For scalar fields we proved that $T_a(y)$ is the dot product of the gradient vector $\nabla f(a)$ with y, For vector fields we will prove that $T_a(y)$ is a vector whose k th component is the dot product $\nabla f_k(a)\cdot y$.

<u>Theorem 1.13.1</u> Assume f is differentiable at a with total derivative T_a. Then the derivative $f'(a;y)$ exists for every a in $\mathbb{R}^n$, and we have

$$T_a(y) = f'(a;y) \quad \ldots \quad (2)$$

Moreover, if $f = (f_1, f_2, \ldots, f_m)$ and if $y = (y_1, y_2, \ldots, y_n)$, we have $T_a(y) = \sum_{k=1}^{m} \nabla f_k(a)\cdot y \, e_k = \left(\nabla f_1(a)\cdot y, \nabla f_2(a)\cdot y, \ldots, \nabla f_m(a)\cdot y\right)$.
$$\ldots \quad (3)$$

Proof: We argue as in the scalar case. If $y = 0$, then $f'(a;y) = 0$ and $T_a(0) = 0$. Therefore we can assume that $y \neq 0$. Taking $u = hy$ in the Taylor formula (1), we have

$$f(a+hy) - f(a) = T_a(hy) + \|hy\| E(a,u)$$
$$= h\,T_a(y) + |h|\,\|y\|\,E(a,u).$$

Dividing by h and letting $h \to 0$, we obtain (2).

To prove (3) we simply note that

$$f'(a;y) = \sum_{k=1}^{m} f_k'(a;y) e_k = \sum_{k=1}^{m} \nabla f_k(a)\cdot y \, e_k .$$

Equation (3) can also be written more simply as a matrix product,
$$T_a(y) = Df(a)\,y,$$
where $Df(a)$ is the $m\times n$ matrix whose k th