can be similarly done.

1.16 Maxima, minima and saddle points

A surface that is described explicitly by an equation of the form $z = f(x,y)$ can be thought of as a level surface of the scalar field F defined by the equation

$$F(x,y,z) = f(x,y) - z,$$

If f is differentiable, the gradient of this field is given by the vector

$$\nabla F = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} - \hat{k}$$

A linear equation for the tangent plane at a point $P_1 = (x_1, y_1, z_1)$ can be written in the form

$$z - z_1 = A(x - x_1) + B(y - y_1)$$

where $A = D_1 f(x_1, y_1)$ and $B = D_2 f(x_1, y_1)$

When both coefficients A and B are zero, the point P_1 is called a stationary point of the surface and the point (x_1, y_1) is called a stationary point or a critical point of the function f. The tangent plane is horizontal at a stationary point. The stationary points of a surface are usually classified into three categories:

maxima, minima and saddle points. If the surface is thought of as a mountain landscape, these categories correspond, respectively, to mountain tops, bottoms of valleys, and mountain passes.

The concepts of maxima, minima and saddle points can be introduced for arbitrary scalar fields defined on subsets of $\mathbb{R}^n$.

Definition: A scalar field f is said to have an absolute maximum at a point a of a set S in $\mathbb{R}^n$ if $\quad f(x) \leq f(a) \quad \cdots \quad (1)$

for all x in S. The number $f(a)$ is called the absolute maximum value of f on S. The function f is said to have a relative maximum at a if the inequality in (1) is satisfied for every x in some n-ball $B(a)$ lying in S.

In other words, a relative maximum at a is the absolute maximum in some neighbourhood of a.

The terms absolute minimum and relative minimum are defined in an analogous fashion, using the inequality opposite to that in (1). The adjectives global and local are sometimes used in place of absolute and relative, respectively.

Definition A number which is either a relative maximum or a relative minimum of f is called an extremum of f.

If f has an extremum at an interior point a and is differentiable there, then all first-order partial derivatives $D_1f(a), D_2f(a), \ldots, D_nf(a)$ must be zero. In other words, $\nabla f(a) = 0$. (This is easily proved by holding each component fixed and reducing the problem to the one dimensional case.) In the case $n=2$, this means that there is a horizontal tangent plane to the surface $z=f(x,y)$ at the point $(a, f(a))$. On the other hand, it is easy to find examples in which the vanishing of all partial derivatives at a does not necessarily imply an extremum at a. This occurs at the so-called saddle points which are defined as follows:

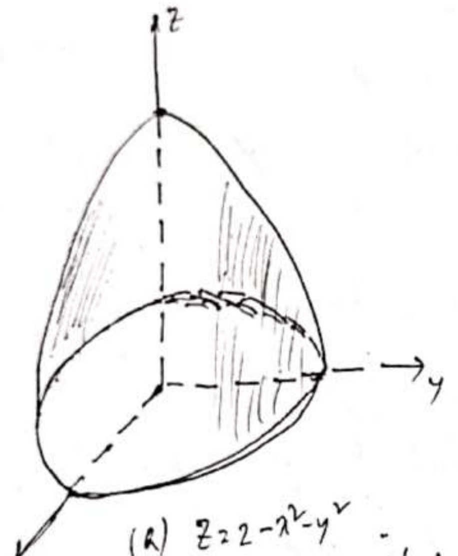
Definition Assume f is differentiable at a. If $\nabla f(a) = 0$ the point a is called a stationary point of f. A stationary point a is called a saddle point if every n-ball contains points x such that $f(x) < f(a)$ and other points such that $f(x) > f(a)$.

The situation is somewhat analogous to the one-dimensional case in which stationary points of a function are classified as maxima, minima and points of inflexion.

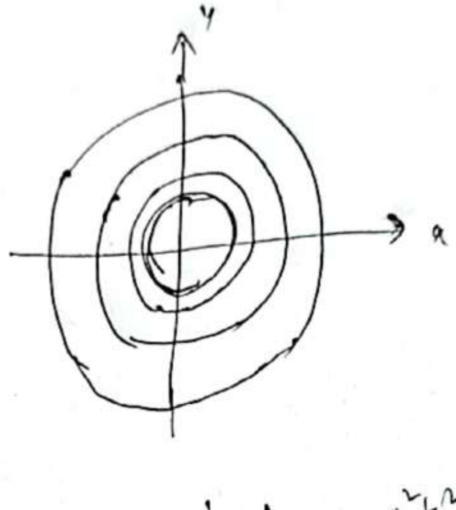
The following examples illustrate several type of stationary points. In each case the stationary point in question is at the origin.

<u>Example 1</u> Relative maximum. $z = f(x,y) = 2 - x^2 - y^2$.

This surface is a paraboloid of revolution. In the vicinity of the origin it has shape shown in Figure 1. Its level curves are circles, some of

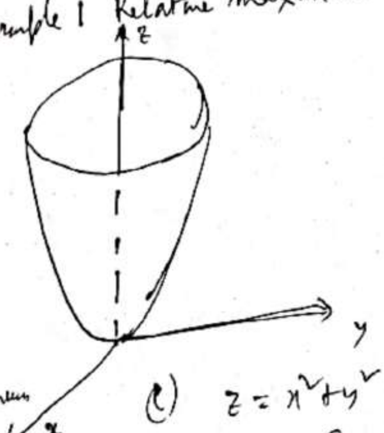


(a) $z = 2 - x^2 - y^2$



(b) Level curves $x^2+y^2=c$

Example 1 Relative maximum at the origin



(c) $z = x^2+y^2$

Example 2 Relative minimum at the origin

Figure 1

which are shown in Figure 1(b). Since $f(x,y) = 2 - x^2 - y^2 = 2 - (x^2+y^2) \leq 2 = f(0,0)$ for all (x,y), it follows that f not only has a relative maximum at $(0,0)$, but also an absolute maximum there. Both partial