derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ vanish at the origin.

Example 2 Relative minimum. $z = f(x,y) = x^2+y^2$. This example, another paraboloid of revolution is essentially the same as Example 1, except that there is a minimum at the origin rather than a maximum. The appearance of the surface near the origin is illustrated in Figure 1(c) and some of the level curves are shown in Figure 1(b).

Example 3. Saddle point. $z = f(x,y) = xy$. This surface is a hyperbolic paraboloid. Near the origin the surface is saddle shaped as shown in Figure 2(a). Both partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are zero at the origin but there is neither a relative maximum nor a relative minimum there. In fact, for points (x,y) in the first or third quadrant, x and y have the same sign, giving $f(x,y) > 0 = f(0,0)$, whereas in the second and 4th quadrant x, y have opposite signs, giving us $f(x,y) < 0 = f(0,0)$. Therefore in every neighbourhood of the origin there are points at which the function is less than $f(0,0)$ and points at which the function exceeds $f(0,0)$, so the origin is a saddle point

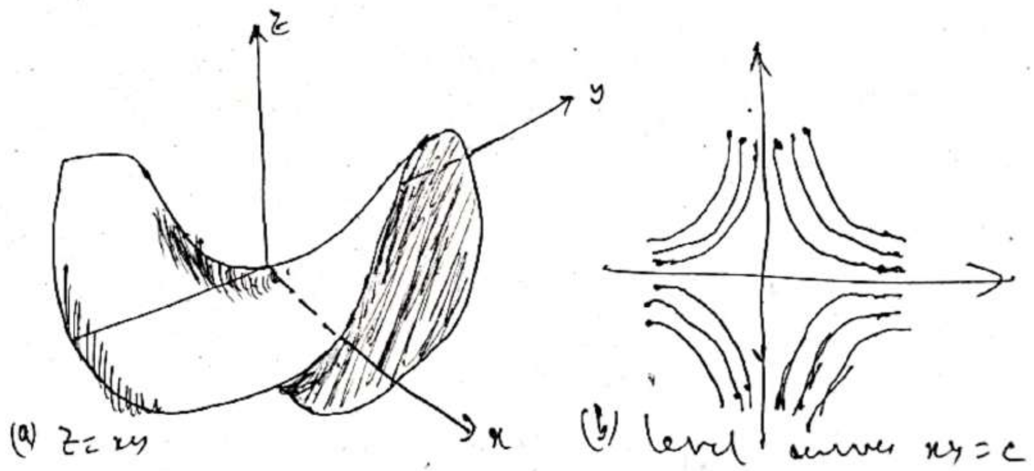


Figure 2 Examples Saddle point at the origin

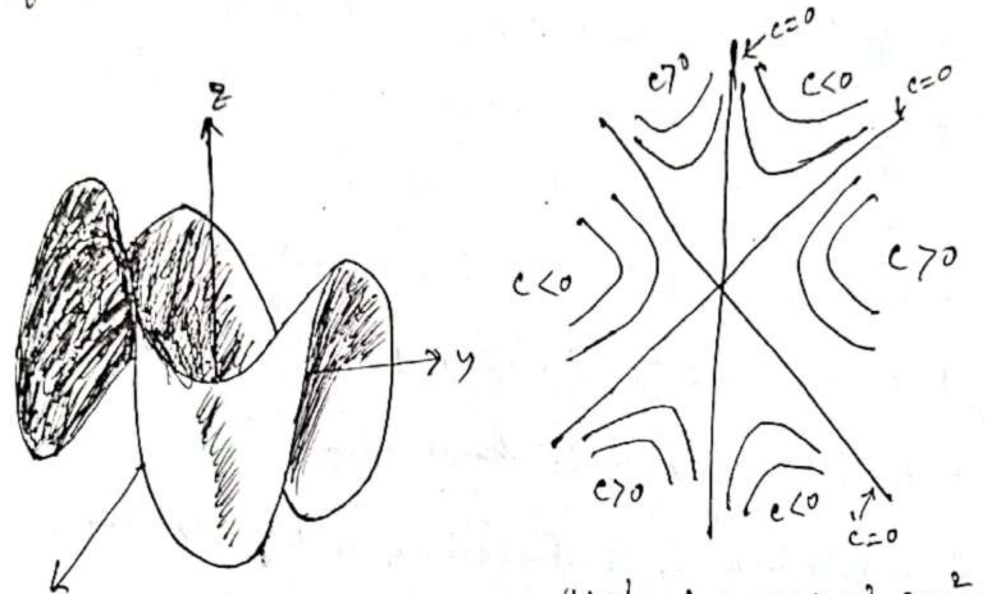
The presence of the saddle point is also ~~not~~ revealed by figure 2(b) which shows some of the level curves near (0,0). These are hyperbolas having the x-axis and y-axis as asymptotes.

<u>Example 4.</u> Saddle point. $z = f(x,y) = x^3 - 3xy^2$. Near the origin, this surface has the appearance of a mountain pass in the vicinity of three peaks. This surface, sometimes referred to as a "monkey saddle", is shown in Figure 3(a). Some of the level curves are illustrated in 3(b). It is clear that there is a saddle point at the origin.

<u>Example 5</u> Relative minimum. $z = f(x,y) = x^2 y^2$. This surface has the appearance of a valley surrounded by four mountains, as suggested by 4(a). There is an absolute minimum at the origin, since $f(x,y) \geq f(0,0)$ for all of x, y. The level curves [shown in Figure 4(b)] are hyperbolas having the x-axis and y-axis as asymptotes. Note that these level curves are similar to those in Example 3. In this case, however the function assumes only non-negative values on all its level curves.

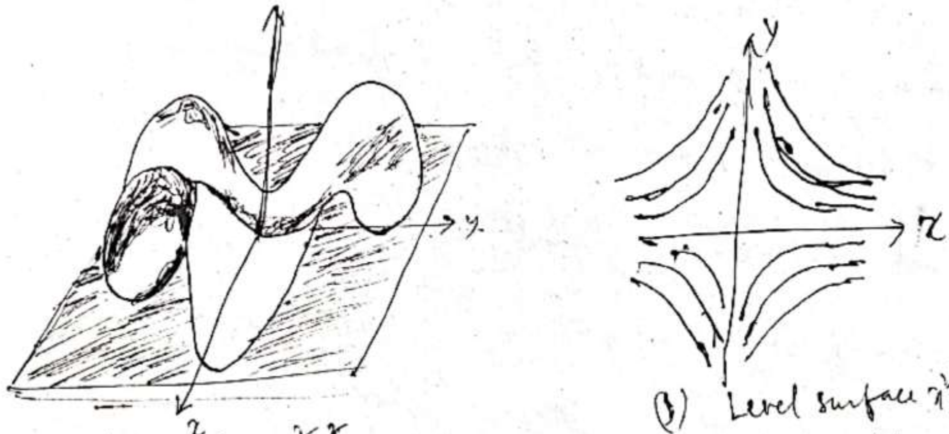
<u>Example 6</u> Relative maximum. $z = f(x,y) = 1 - x^2$. In this case surface is a cylinder with generators parallel to y-axis as shown in 5(a). Cross section cut by

planes parallel to the z-axis are parabolas. There is obviously an absolute maximum at the origin because $f(x,y) = 1-x^2 \leq 1 = f(0,0)$ for all (x,y). The level curves form a family of parallel straight lines as shown in 5(b)



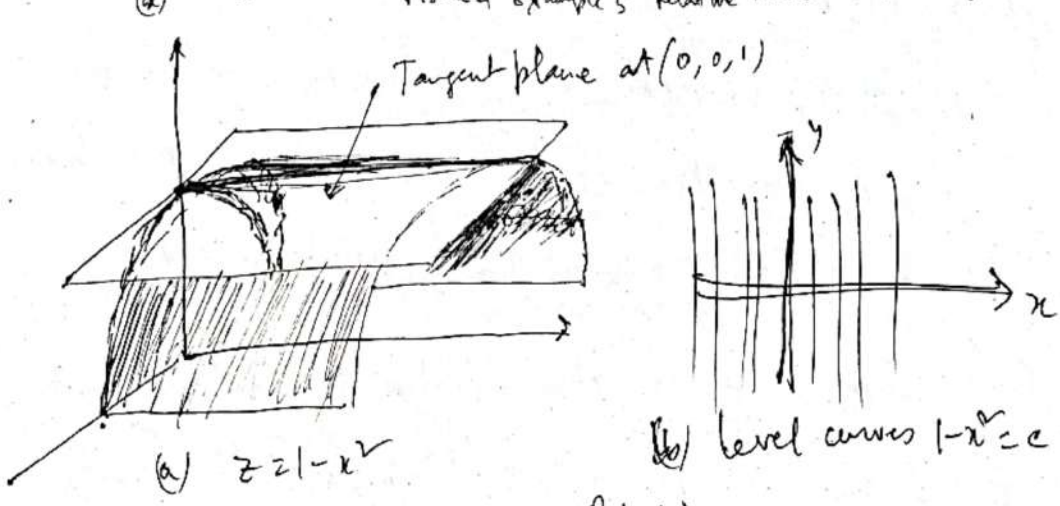
(a) $z = x^3 - 3xy^2$. (b) level curves: $x^3 - 3xy^2 = c$

Figure 3 Example 4. Saddle point at the origin.



(a) $z = x^2y^2$ (b) Level surface $x^2y^2 = c$

Figure 4 Example 5 Relative minimum at the origin



(a) $z = 1 - x^2$ (b) level curves $1 - x^2 = c$

Figure 5 Example 6 Relative maximum at origin

1.17 Extrema (or Extreme values) of functions of two variables :

Definition: Let $S \subseteq \mathbb{R}^2$ and $f: S \to \mathbb{R}$ be a function of two independent variables x and y. Let (a,b) be an interior point of S. f is said to have an extreme (or local extreme) value at (a,b) or equivalently $f(a,b)$ is said to be an extreme value of f, if there exists a suitable neighbourhood $N(a,b)$ of (a,b) such that for all $(x,y) \in N(a,b)$,

$f(x,y) - f(a,b)$ does not change sign.

f is said to have a (local) maximum or a (local) minimum value at (a,b) according as $f(x,y) - f(a,b) \leq 0$ or ≥ 0 for all $(x,y) \in N(a,b)$.

Remark: f is said to have a global maximum at $(a,b) \in S$ if $f(x,y) - f(a,b)$ does not change sign in S. f has a global maximum or global minimum at (a,b) according as $f(x,y) - f(a,b) \leq 0$. or ≥ 0 for all (x,y) in S.

Theorem 1.17.1 (Necessary condition for existence of extreme value) : Let $S \subset \mathbb{R}^2$, $f: S \to \mathbb{R}$ and (a,b) be an interior point of S. If partial derivative f_x, f_y exists at (a,b) and f has an extreme value at (a,b), then $f_x(a,b) = 0$, $f_y(a,b) = 0$

Proof: ~~Since~~ f has an extreme value at (a,b) implies both the function of single variable $f(x,b)$ and $f(a,y)$ have.