extreme values at $x=a$ and $y=b$ respectively. Since f_x and f_y exists at (a,b), by a necessary condition for the extreme value of a function of single variable $f_x(x,b)$ vanishes at $x=a$ and $f_y(a,y)$ vanishes at $y=b$. Thus $f_x(a,b)=0$ and $f_y(a,b)=0$ if f has an extreme value at (a,b).

So, the condition is necessary.

Example 1 Show that the function $f(x,y)=|x|+|y|$, $(x,y)\in\mathbb{R}^2$ possesses an extreme value (minimum value) at $(0,0)$ although $f_x(0,0)$ and $f_y(0,0)$ do not exist.

$$f_x(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{|h|}{h} = 1 \text{ or } -1$$

according as $h>0$ or $h<0$. So $\lim_{h\to 0}\frac{|h|}{h}$ does not exist, so $f_x(0,0)$ does not exist.

Similarly $f_y(0,0)$ does not exist

But $f(x,y) = |x|+|y| \geq f(0,0)=0$ for every point (x,y) in any neighbourhood of $(0,0)$ confirming that f has a minimum value at $(0,0)$

We repeat the definition of stationary point and saddle point once more

Definition Let $S\subseteq\mathbb{R}^2$ and $f:S\to\mathbb{R}$. An interior point (a,b) of S is said to be a stationary point of f in S if both $f_x(a,b)$ and $f_y(a,b)$ exists and $f_x(a,b)=0=f_y(a,b)$, Example 2 Find all the stationary points of

$$f(x,y) = 4x^2 - xy + 4y^2 + x^3y + xy^3 - 4$$

Solution: Since f is a polynomial in x and y, f possesses partial derivative f_x and f_y at every point (x,y) of its domain $\mathbb{R}^2$.

$$f_x = 8x - y + 3x^2y + y^3$$

$$f_y = -x + 8y + x^3 + 3xy^2$$

$f_x = 0$ and $f_y = 0 \Rightarrow (x+y)^3 + 7(x+y) = 0$ (adding both)

$\Rightarrow (x+y)\left((x+y)^2+7\right) = 0 \Rightarrow x+y=0$, since $(x+y)^2 \geq 0$

Then $f_x = 0$ and $x+y=0 \Rightarrow 9x - 4x^3 = 0$

$\Rightarrow x(9-4x^2) = 0 \Rightarrow x = 0$ or $x = \pm \frac{3}{2}$.

Since $x+y=0$ so $y=0$ or $y = \mp \frac{3}{2}$.

Thus $(0,0)$, $\left(\frac{3}{2}, -\frac{3}{2}\right)$ and $\left(-\frac{3}{2}, \frac{3}{2}\right)$ are the three stationary points of the function $f(x,y)$

Definition: An interior point (a,b) of the domain of a function f of two independent variables x and y is said to be a saddle point of f if it is a stationary point of f i.e., if $f_x(a,b)=0$, $f_y(a,b)=0$ but f has neither maximum nor minimum value at (a,b).

Example 3 Show that $(0,0)$ is a saddle point of $f(x,y) = x^6 + (x-y)^3$

$f_x(x,y) = 6x^5 + 3(x-y)^2, \quad f_y(x,y) = -3(x-y)^2.$

At $(0,0)$ both f_x, f_y vanish, so $(0,0)$ is a ~~sad~~ stationary point of f.

Now in any neighbourhood of $(0,0)$, there are points where $x>y$ and there are points where $x<y$. Thus $x-y$ is positive for some points and negative for some other points in any neighbourhood of $(0,0)$. Since $x^6+(x-y)^3 > 0$ when $x>y$ and $x^6+(x-y)^3<0$ when $y>0$ and $x=0$, which shows that $f(x,y)-f(0,0)$ does change sign in any neighbourhood of $(0,0)$. This proves that f has no extreme value at $(0,0)$ and hence $(0,0)$ is a saddle point of f.

<u>Example 4</u> Let $f(x,y) = \begin{cases} 0, & xy=0 \\ 1, & xy \neq 0 \end{cases}$

Examine whether $(0,0)$ is a point where f attains its extreme value

solution: $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{0-0}{h} = 0$

$f_y(0,0) = \lim\limits_{k\to 0} \dfrac{f(0,k)-f(0,0)}{k} = \lim\limits_{k\to 0} \dfrac{0-0}{k} = 0$

$\Rightarrow (0,0)$ is a stationary point of f

$f(x,y) - f(0,0) \geq 0$ for all (x,y) in any neighbourhood of $(0,0)$. Hence f has a minimum at $(0,0)$.

5. Show that the function $f(x,y) = (x-y)^3 + (2-x)^2$ has a saddle point at $(2,2)$

Solution: $f_x(x,y) = 3(x-y)^2 - 2(2-x)$, $f_y(x,y) = -3(x-y)^2$

~~At $(2,2)$ $f_x(x,y) = 0$~~

Now $f_x(2,2) = 0$ and $f_y(2,2) = 0$. Thus $(2,2)$ is a stationary point of f. For any point $(2+h, 2+k)$ in any neighbourhood of $(2,2)$

$$f(2+h, 2+k) - f(2,2) = (h-k)^3 + h^2$$

For $h=0, k>0$, the difference is negative and for $h>0$, $k=0$ the difference is positive.

Hence $f(2+h, 2+k) - f(2,2)$ does change sign in any neighbourhood of $(2,2)$. This proves that $(2,2)$ is a saddle point of f.

Theorem 1.17.2 (Sufficient condition for the existence of extreme value):

Let $S \subset \mathbb{R}^2$ and $f: S \to \mathbb{R}$ be a function of two independent variables x and y. Let (a,b) be an interior point of S such that $f_x(a,b) = 0$ and $f_y(a,b) = 0$. Let f ~~possesses~~ possesses continuous second order partial derivatives in a certain neighbourhood $N(a,b)$ of (a,b) such that $f_{xx}(a,b)$, $f_{xy}(a,b)$ and $f_{yy}(a,b)$ are not all zero. Then