$B(p, \delta_i) \subset G_i$, $i = 1, 2, \ldots, n$. Setting $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_n\}$, we get $B(p, \delta) \subset \bigcap_{i=1}^{n} G_i$. So, every $p \in \bigcap_{i=1}^{n} G_i$ is its interior point.

Hence $\bigcap_{i=1}^{n} G_i$ is open

Result 1.2.4 The intersection of an arbitrary family of closed sets in $\mathbb{R}^2$ is ~~an~~ a closed set in $\mathbb{R}^2$.

Proof : Let $\{F_\alpha : \alpha \in I\}$ be an arbitrary family of closed sets in $\mathbb{R}^2$. Let $F = \bigcap_{\alpha \in I} F_\alpha$

Now $F^c = \left(\bigcap_{\alpha \in I} F_\alpha\right)^c = \bigcup_{\alpha \in I} F_\alpha^c = \bigcup_{\alpha \in I} G_\alpha$ where $G_\alpha = F_\alpha^c$

and G_α is open for each $\alpha \in I$.

So, $\bigcup_{\alpha \in I} G_\alpha$ is open. Hence F^c is open

Hence $F = (F^c)^c$ is closed in $\mathbb{R}^2$

So, $\bigcap_{\alpha \in I} F_\alpha$ is open in $\mathbb{R}^2$.

<u>Result 1.2.5</u> The union $\bigcup_{i=1}^{n} F_i$ of a finite number of closed sets F_i, $i = 1, 2, \ldots, n$ is a closed set in $\mathbb{R}^2$.

<u>Proof:</u> $\left(\bigcup_{i=1}^{n} F_i\right)^c = \bigcap_{i=1}^{n} F_i^c = \bigcap_{i=1}^{n} G_i$ where $G_i = F_i^c$ is open in $\mathbb{R}^2$, $i = 1, 2, \ldots, n$. By result 1.2.3, $\bigcap_{i=1}^{n} G_i$ is open in $\mathbb{R}^2$. So $\left(\bigcup_{i=1}^{n} F_i\right)^c$ is open. Hence $\bigcup_{i=1}^{n} F_i = \left(\left(\bigcup_{i=1}^{n} F_i\right)^c\right)^c$ is closed in $\mathbb{R}^2$.

Result 1.2.6 F is closed in $\mathbb{R}^2 \Leftrightarrow F = \overline{F}$ in $\mathbb{R}^2$

Proof: Let F be closed in $\mathbb{R}^2$. Let $x \in \mathbb{R}^2$ and $x \notin F$. Then the open set $G = \mathbb{R}^2 \setminus F$ is a neighbourhood of x that contains no points of F. So we have the result that if $x \notin F$, x is not a limit point of F. So, $F = \overline{F}$

Conversely, let $F = \overline{F}$. We propose to show that $G = \mathbb{R}^2 \setminus F$ is open set in $\mathbb{R}^2$. If $x \in G$, then $x \notin \overline{F}$ and therefore, x is not a limit point of F. Hence there exists neighbourhood of x containing only a finite number of points $x_1, x_2, \ldots, x_n$ of F. Since $x \notin F$, we can construct discs about x, $O_1(x), O_2(x), \ldots, O_n(x)$ such that $x_i \notin O_i(x)$, $i = 1, 2, \ldots, n$. Then $O(x) = \bigcap_{i=1}^{n} O_i(x)$ is an open neighbourhood of x containing no points of F at all, $O(x) \subset \mathbb{R}^2 \setminus F$ and hence $\mathbb{R}^2 \setminus F = \mathbb{R}^2 \setminus \overline{F}$ is open. Therefore, F is closed in $\mathbb{R}^2$

Note 1. The set $M \subset \mathbb{R}^2$ is a neighbourhood of $(a, b) \in \mathbb{R}^2$ if and only if $\exists$ a $\delta > 0$ such that

$$N((a, b), \delta) \subset M$$

Note 2. All these results can be generalised in $\mathbb{R}^n$.

1.2.7. Functions from $\mathbb{R}^n (n > 1)$ to $\mathbb{R}^m (m \geq 1)$

Here we take $\mathbb{R}^n (n > 1)$ and $\mathbb{R}^m (m \geq 1)$ as the real euclidean space with standard inner product and the induced norm and distance. For both of $\mathbb{R}^n$ and $\mathbb{R}^m$ we shall use the same notation for norms.

We first give some examples of functions from $\mathbb{R}^n$ $(n \geq 1)$ to $\mathbb{R}^m$ $(m \geq 1)$

Examples 1. $f: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \sin(x^2 y)$$

or, $$f(x,y) = \log(x^2+y^2)$$

or, $$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

2. $f: \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$f(x,y,z) = (x^2, y^2, z^2, x^2+y^2)$$

or, $$f(x,y,z) = (z, y, x, 1)$$

or, $$f(x,y,z) = (\sin x, \sin y, \sin z, x+y+z)$$

3. $f: \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x,y,z) = x+y+z$$

or, $$f(x,y,z) = \sin x + \sin y + \sin z$$

or, $$f(x,y,z) = x^2 y z$$

1.2.8 Limits and continuity

We consider a function $f: S \to \mathbb{R}^m$, $S \subset \mathbb{R}^n$. If $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ we write $\lim_{x \to a} f(x) = b$

(or, $f(x) \to b$ as $x \to a$)

to mean that $$\lim_{\|x-a\| \to 0} \|f(x) - b\| = 0 \quad \cdots \quad (1)$$

The limit symbol in equation (1) is the usual limit of elementary calculus.

i.e., $\lim_{x \to a} f(x) = b$ if given an $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that $\|f(x) - b\| < \varepsilon$ for $0 < \|x-a\| < \delta$

In this definition it is not required that f be defined at the point a itself.

If we write $h = x - a$, equation (1) becomes

$$\lim_{\|h\| \to 0} \|f(a+h) - b\| = 0$$

For points in $\mathbb{R}^2$, we write (x_1, x_2) for x and (a_1, a_2) for a and express the limit as $\lim_{(x_1, x_2) \to (a_1, a_2)} f(x_1, x_2) = b$

For points in $\mathbb{R}^3$, write (x_1, x_2, x_3) for x and (a_1, a_2, a_3) for a and express the limit as $\lim_{(x_1, x_2, x_3) \to (a_1, a_2, a_3)} f(x_1, x_2, x_3) = b$

A function $f : S \to \mathbb{R}^m$, $S \subseteq \mathbb{R}^n$ is said to be continuous at a point $a \in \mathbb{R}^n$ if

$$\lim_{x \to a} f(x) = f(a)$$

We say that f is continuous on the set S if f is continuous at each point of S.

Since these definitions are straightforward extensions