(i) f has no extreme value if

$$H = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{vmatrix} < 0$$ (H is called the Hessian matrix)

(ii) f has an extreme value if $H > 0$ and in particular f has a maximum value at (a,b) if $f_{xx}(a,b) < 0$ or $f_{yy}(a,b) < 0$ and f has a minimum value at (a,b) if $f_{xx}(a,b) > 0$ or $f_{yy}(a,b) > 0$

(iii) If $H = 0$ then f may or may not have extreme value at (a,b).

~~Proof. Let $(a+h, b+k)$ be any point in the neighbourhood of (a,b). Then from Taylor's~~

Proof is omitted here i.e., we are stating the above theorem without proof.

Working rule to examine the extreme values of function of two variables:

Step 1 Find (a,b) such that $f_x(a,b) = 0$ and $f_y(a,b) = 0$

Step 2 Find the principal minor of the matrix $\begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{bmatrix}$ of the quadratic

form d^2f = ~~...~~
$$= A\,dx^2 + 2B\,dx\,dy + C\,dy^2$$

where $A = f_{xx}(a,b)$, $B = f_{xy}(a,b)$ and $C = f_{yy}(a,b)$

and $H = \begin{vmatrix} A & B \\ B & C \end{vmatrix}$

~~Step~~ Step 3. If $A>0$, $H>0$ then at (a,b), d^2f is positive definite and hence f has a minimum at (a,b)

If $A<0$, $H>0$ then at (a,b), d^2f is negative definite and hence f has a maximum at (a,b)

If $H<0$ then at (a,b), d^2f is indefinite, i.e., it is neither positive nor negative definite. and Hence f has no extrema at (a,b) and (a,b) is a saddle point of f.

If $H=0$ and $A>0$ then at (a,b) d^2f is positive semi-definite and if $H=0$, $A<0$ then at (a,b), d^2f is negative semi-definite. In these cases no definite conclusion can be drawn at this stage and one has to use the definition of extreme value of a function to examine whether f has extreme value or not at (a, b).

Worked Examples

Example 1 Let $f(x, y) = x^4 + y^4 - 2x^2$. Show that f has a local minimum value at $(-1, 0)$ and $(1, 0)$ and $(0, 0)$ is a saddle point of f.

Solution: $f_x = 4x(x^2 - 1)$, $f_y = 4y^3$, f_x, f_y vanish at $(0,0), (1,0), (-1,0)$. Hence these three points are stationary points of f.

$f_{xx} = 12x^2 - 4 \quad f_{xy} = 0$

$f_{yy} = 12y^2$

So, $f_{xx}(1,0) = 4$, $f_{xy}(1,0) = 0$, $f_{yy} = 0$

Here $f_{xx}(1,0) > 0$ and $\begin{vmatrix} f_{xx}(1,0) & f_{xy}(1,0) \\ f_{xy}(1,0) & f_{yy}(1,0) \end{vmatrix}$

$$= \begin{vmatrix} 4 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

So no conclusion at this stage can be drawn

Also $f_{xx}(-1,0) = 4$, $f_{xy}(-1,0) = 0$, $f_{yy}(-1,0) = 0$

So, $H = \begin{vmatrix} 4 & 0 \\ 0 & 0 \end{vmatrix} = 0$

So, no conclusion can be drawn,

now $f(1+h, 0+k) - f(1,0)$

$= (1+h)^4 + k^4 - 2(1+h)^2 + 1$

$= \left((1+h)^2 - 1\right)^2 + k^4 > 0$ in any neighbourhood of $(1,0)$ since h, k are not simultaneously zero.

Hence f has a minimum at $(1,0)$

$f(-1+h, 0+k) - f(-1,0)$

$= (-1+h)^4 + k^4 - 2(-1+h)^2 + 1$

$= \left((-1+h)^2 - 1\right)^2 + k^4 > 0$ in any neighbourhood of $(-1,0)$ since h, k are not zero at the same time.

Thus f has a local minimum at $(-1,0)$

For the point $(0,0)$

$f_{xx}(0,0) = -4$, $f_{xy}(0,0) = 0$, $f_y(0,0) = 0$

$f(0+h, 0+k) - f(0,0) = h^4 + k^4 - 2h^2 = h^2(h^2-2) + k^4$

For (h,k) in any neighbourhood of $(0,0)$,

when $0 < |h| < \sqrt{2}$, $k = 0$, $f(h,k) - f(0,0) < 0$

and when $h = 0$, $k \neq 0$, $f(h,k) - f(0,0) > 0$

Thus $f(h,k) - f(0,0)$ does change sign in any neighbourhood of $(0,0)$, which implies f has no extreme value at $(0,0)$, i.e., $(0,0)$ is a saddle point.