Example 2 Show that if $f(x,y) = 2x^4 - 3x^2y + y^2$ then $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(0,0)$ but f has neither a maximum nor a minimum value at $(0,0)$.

Solution: $f_x(x,y) = 8x^3 - 6xy \quad f_y(x,y) = -3x^2 + 2y$

$f_{xx}(x,y) = 24x^2 - 6y \quad f_{xy}(x,y) = -6x \quad f_{yy}(x,y) = 2$

Clearly, $f_x(0,0) = 0 = f_y(0,0)$

$f_{xx}(0,0) = 0, \; f_{xy}(0,0) = 0, \; f_{yy}(0,0) = 2$

Hence $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(0,0)$

Now $f(x,y) - f(0,0) = 2x^4 - 3x^2y + y^2$

$$= (2x^2 - y)(x^2 - y) > 0 \quad \text{for } y < 0 \text{ or } x^2 > y > 0$$
$$< 0 \quad \text{for } y > x^2 > \frac{y}{2} > 0$$

Hence $f(x,y) - f(0,0)$ does change sign in any neighbourhood of origin. Hence f has neither maximum nor minimum at $(0,0)$.

Example 3 Show that for the function f, where $f(x,y) = y^2 + 2x^2y + 2x^4$, $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(0,0)$ and f has a minimum value at $(0,0)$.

Solution: $f(x,y) = y^2 + 2x^2y + 2x^4$

$f_x(x,y) = 4xy + 8x^3, \quad f_y(x,y) = 2y + 2x^2$

Since $f_x(0,0) = 0$, $f_y(0,0) = 0$, $(0,0)$ is a stationary point of f.

$$f_{xx}(x,y) = 4y + 24x^2 \quad f_{xy}(x,y) = 4x, \; f_{yy}(x,y) = 2$$

Hence $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(0,0)$

now $f(x,y) - f(0,0) = (y+x^2)^2 + x^4 \geq 0$ in any neighbourhood of $(0,0)$.

Hence, from the definition, f has a minimum at $(0,0)$

Example 4 Let $f(x,y) = xy \log_e(x^2+y^2)$ which is defined everywhere in the plane $\mathbb{R}^2$ except at the origin.

Justify the following:

(i) the points $(0,\pm 1)$ and $(\pm 1, 0)$ are not extrema of the function

(ii) $\left(\frac{1}{\sqrt{2e}}, \frac{-1}{\sqrt{2e}}\right)$ and $\left(\frac{-1}{\sqrt{2e}}, \frac{1}{\sqrt{2e}}\right)$ are local maxima of the function

Solution: Here $f_x = y \log_e(x^2+y^2) + \frac{2x^2y}{x^2+y^2}$,

$$f_y = x\log_e(x^2+y^2) + \frac{2xy^2}{x^2+y^2}$$

We note that both f_x and f_y vanish at all the points $(0,\pm 1), (\pm 1, 0), \left(\frac{1}{\sqrt{2e}}, \frac{-1}{\sqrt{2e}}\right), \left(\frac{-1}{\sqrt{2e}}, \frac{1}{\sqrt{2e}}\right)$

$$f_{xx} = \frac{2xy(x^2+3y^2)}{(x^2+y^2)^2}, \quad x^2+y^2 \neq 0$$

$$f_{yy} = \frac{2xy(3x^2+y^2)}{(x^2+y^2)^2}, \quad x^2+y^2 \neq 0,$$

$f_{xy} = \log_e(x^2+y^2) + \dfrac{2(x^4+y^4)}{(x^2+y^2)^2}$, $x^2+y^2 \neq 0$

At $(0, \pm 1)$ and $(\pm 1, 0)$, $f_{xx} f_{yy} - f_{xy}^2 = -4 < 0$

These are saddle points.

At $\left(\frac{\pm 1}{\sqrt{2e}}, \frac{\mp 1}{\sqrt{2e}}\right)$, $f_{xx} f_{yy} - f_{xy}^2 = (-2)(-2) - 0 = 4 > 0$

$f_{xx}\left(\frac{1}{\sqrt{2e}}, \frac{-1}{\sqrt{2e}}\right) = -2 = f_{xx}\left(\frac{-1}{\sqrt{2e}}, \frac{1}{\sqrt{2e}}\right) < 0$

So, local maxima exist at these two points.

Example 5 Find a point within a triangle such that the sum of the squares of its distances from the vertices is a minimum.

Solution: Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be the vertices of the triangle.

Here $f(x,y) = (x-x_1)^2 + (y-y_1)^2 + (x-x_2)^2 + (y-y_2)^2 + (x-x_3)^2 + (y-y_3)^2$

$f_x = 2\left[3x - (x_1+x_2+x_3)\right]$, $f_y = 2\left[3y - (y_1+y_2+y_3)\right]$

$f_x = 0 = f_y \Rightarrow x = \dfrac{x_1+x_2+x_3}{3}$, $y = \dfrac{y_1+y_2+y_3}{3}$

$f_{xx} = 6 = f_{yy}$, $f_{xy} = 0$

At the above point $f_{xx} f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} = 6 > 0$

So, $f(x,y)$ is a minimum for the above point which is the centroid of the triangle.

Example 6 (Least Square Line) Let the n points

$P_i(x_i, y_i)$, $i=1,2,\ldots,n$ obtained by observation or experiment, are theoretically points of a straight line, but due to slight errors in measurement, fails to have the property. What is the 'best' straight line through the given points in the sense of least squares?

Solution: Let $y = ax+b$ $(a, b \in \mathbb{R})$ be the required line. So, the given problem reduces to

determine a, b so that

$$f(a,b) = \sum_{i=1}^{n} (ax_i + b - y_i)^2$$

is a minimum

Here f_a and f_b both exist.

So, $f_a = 0 = f_b$ give $2\sum_{i=1}^{n} x_i(ax_i + b - y_i) = 0$ --- (1)

and $2\sum_{i=1}^{n}(ax_i + b - y_i) = 0$ --- (2)

It can be shown that $\dfrac{\sum x_i^2}{n} > \left(\dfrac{\sum_{i=1}^{n} x_i}{n}\right)^2$ --- (3)

where x_i's are not always same. So, (1) and (2) can be solved uniquely for a and b

Equation (2) $\Rightarrow \dfrac{a\left(\sum_{i=1}^{n} x_i\right)}{n} + b = \dfrac{\left(\sum_{i=1}^{n} y_i\right)}{n}$

It is customary to take $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$

and $(\bar{x}, \bar{y})$ is designated as mean centre.