So, $a\bar{x} + b = \bar{y}$

$$f_{aa} = 2\sum_{i=1}^{n} x_i^2, \quad f_{ab} = 2\sum_{i=1}^{n} x_i, \quad f_{bb} = 2n$$

So (3) $\Rightarrow$ $f_{aa} f_{bb} - f_{ab}^2 > 0$. Also $f_{aa} > 0$

Thus ~~f(a,b)~~ minimum of $f(a,b)$ is assured, for the a, b obtained from (1) and (2).

Note 1. We can similarly check for maxima and minima of function of three variables.

Let (a,b,c) be such that $f_x = f_y = f_z = 0$ at (a,b,c)

At that point, let

$$D_1 = f_{xx}, \quad D_2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}, \quad D_3 = \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix}$$

(here $f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$, $f_{yz} = f_{zy}$)

Then (i) f is maximum at (a,b,c)
if $D_1 < 0$, $D_2 > 0$, $D_3 < 0$

(ii) f is minimum at (a,b,c)
if $D_1 > 0$, $D_2 > 0$, $D_3 > 0$

The proofs follows from the theory of quadratic forms.

In fact, here

$$d^2 f = f_{xx}\,dx^2 + f_{yy}\,dy^2 + f_{zz}\,dz^2 + 2f_{xy}\,dx\,dy + 2f_{yz}\,dy\,dz + 2f_{zx}\,dz\,dx.$$

$d^2 f$ is negative definite or positive definite according to conditions ~~and~~ stated in (i) and (ii).

If $D_2 < 0$ then $d^2 f(a,b,c)$ is indefinite and hence f has no extreme value at (a,b,c).

If $D_2 = 0$, then nothing can be concluded at this stage and further investigation is necessary.

Example 1 Examine for existence of maxima/minima of the function $f(x,y,z) = x^2+y^2+3z^2-xy+2zx+yz$

Solution: $f_x = 2x-y+2z$, $f_y = 2y-x+z$ and

$f_z = 6z+2x+y$

Since the coefficient matrix $\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 6 \end{bmatrix}$ ~~is non-singular~~ of the system $f_x=0, f_y=0$ and $f_z=0$ is non-singular, so $(0,0,0)$ is the only solution of $f_x=0, f_y=0$ and $f_z=0$

So, $(0,0,0)$ is the only stationary point of f.

$f_{xx}(0,0,0) = 2$, $f_{yy}(0,0,0) = 2$, $f_{zz}(0,0,0) = 6$

$f_{yx}(0,0,0) = f_{xy}(0,0,0) = -1$ $f_{yz}(0,0,0) = f_{zy}(0,0,0) = 1$,

$f_{zx}(0,0,0) = f_{xz}(0,0,0) = 2$

Hence at $(0,0,0)$ $f_{xx} > 0$, $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0$

$$\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} 2 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 6 \end{vmatrix} = 22-8-10 = 4 > 0$$

Hence $d^2f(0,0,0)$ is positive definite. Therefore f has a minimum value at $(0,0,0)$.

Example 2 Show that the function

$$f(x,y,z) = 3\log_e(x^2+y^2+z^2) - 2(x^3+y^3+z^3), \quad (x,y,z) \neq (0,0,0)$$

has only one extreme value, $\log_e\left(\frac{3}{e^2}\right)$.

Solution: $f_x = \dfrac{6x}{x^2+y^2+z^2} - 6x^2$,

$$f_y = \frac{6y}{x^2+y^2+z^2} - 6y^2, \quad f_z = \frac{6z}{x^2+y^2+z^2} - 6z^2.$$

Since $x^2+y^2+z^2 \neq 0$, f_x, f_y, f_z vanish at a point

where $x = y = z = \dfrac{1}{x^2+y^2+z^2}$, $x^2+y^2+z^2 \neq 0$

If $x = y = z = \alpha$, then $3\alpha^3 = 1 \Rightarrow \alpha = \dfrac{1}{\sqrt[3]{3}}$

Thus $\left(\dfrac{1}{\sqrt[3]{3}}, \dfrac{1}{\sqrt[3]{3}}, \dfrac{1}{\sqrt[3]{3}}\right)$ is the only stationary point of f

At (α,α,α), $f_{xx} = \dfrac{6\Sigma\alpha^2 - 12\alpha^2}{(\Sigma\alpha^2)^2} - 12\alpha$

$$= \frac{2}{3\alpha^2} - 12\alpha = \frac{-10}{3\alpha^2} < 0 \quad \left(\text{since } \alpha^3 = \frac{1}{3}\right)$$

Similarly, $f_{yy} = \dfrac{-10}{3\alpha^2}$, $f_{zz} = \dfrac{-10}{3\alpha^2}$

$$f_{xy} = \frac{-12\alpha^2}{(3\alpha^2)^2} = \frac{-4}{3\alpha^2} = f_{yx} = f_{yz} = f_{zy} = f_{xz} = f_{zx}$$

Thus at (α,α,α), $f_{xx} < 0$,

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \frac{84}{9\alpha^4} > 0$$

and

$$\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = -\frac{1}{\alpha^6} \begin{vmatrix} 10 & 4 & 4 \\ 4 & 10 & 4 \\ 4 & 4 & 10 \end{vmatrix}$$

$$= \frac{-632}{\alpha^6} < 0$$

Hence at (α, α, α) $d^2 f$ is negative definite and f has maximum value at $\left(\frac{1}{\sqrt[3]{3}}, \frac{1}{\sqrt[3]{3}}, \frac{1}{\sqrt[3]{3}}\right)$

Maximum Value of f is $3\log_e(3\alpha^2) - 6\alpha^3$

$$= 3\log_e\left(\frac{3}{3^{2/3}}\right) - 2 = 3\log_e 3^{1/3} - 2$$

$$= \log_e 3 - 2 = \log_e\left(\frac{3}{e^2}\right)$$

Method of Lagrange multiplier

Lagrange described a method of determining the extreme values of function of a number of variables which satisfy certain constraints. To discuss his method, we consider the general case of function of $(n+m)$ variables subject to m given constraints.

Let $f(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m)$ be a function of $(n+m)$ variables whose extreme values, if