3. On the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$, find the points whose distances are ~~least~~ least and greatest from the straight line $3x+y-9=0$.

Solution: Let (x,y) be any point on the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$.

Then square of its distance from the straight line $3x+y-9=0$ is $D=\frac{(3x+y-9)^2}{10}=\frac{1}{10}f(x,y)$

where $f(x,y)=(3x+y-9)^2$. We construct the Lagrangian function $L(x,y)=f(x,y)+\lambda\left(\frac{x^2}{4}+\frac{y^2}{9}-1\right)$, where λ is a Lagrange's multiplier constant.

For stationary point, $L_x=0$, $L_y=0$

i.e., $6(3x+y-9)+\frac{\lambda x}{2}=0$ and $2(3x+y-9)+\frac{2\lambda y}{9}=0$

Since stationary point (x,y) is not a point on the straight line $3x+y-9=0$, hence

$$\frac{\lambda x}{4}=\frac{\lambda y}{3} \Rightarrow \frac{x}{4}=\frac{y}{3}=k \text{ (say)}$$

As (x,y) is a point on the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$, we have $k=\pm\frac{1}{\sqrt{5}}$. Hence $\left(\frac{4}{\sqrt{5}},\frac{3}{\sqrt{5}}\right)$ and $\left(-\frac{4}{\sqrt{5}},-\frac{3}{\sqrt{5}}\right)$ are the stationary points and corresponding values of λ are $9(3\sqrt{5}-5)$ and $-9(3\sqrt{5}+5)$.

Since $dL=L_x dx+L_y dy$

$$d^2L=L_{xx}(dx)^2+2L_{xy}\,dx\,dy+L_{yy}(dy)^2$$

At $\left(\frac{4}{\sqrt{5}},\frac{3}{\sqrt{5}}\right)$, $L_{xx}=18+\frac{9}{2}(3\sqrt{5}-5)$, $L_{xy}=6$

$L_{yy}=2+2(3\sqrt{5}+5)$ and from $\frac{x^2}{4}+\frac{y^2}{9}=1$

we have $\frac{x}{4}dx + \frac{y}{9}dy = 0 \Rightarrow dy = -\frac{9x}{4y}dx = -3dx$ at $\left(\frac{4}{\sqrt5}, \frac{3}{\sqrt5}\right)$. Hence at $\left(4/\sqrt5, \frac{3}{\sqrt5}\right)$

$$d^2L = \left[18 + \frac{9}{2}(3\sqrt5 - 5) - 36 + 9\{2 + 2(3\sqrt5+5)\}\right](dx)^2$$

$$= \left(\frac{135}{2} + \frac{135\sqrt5}{2}\right)(dx)^2 > 0$$

Hence d^2L is positive definite which implies f has minimum value at $\left(\frac{4}{\sqrt5}, \frac{3}{\sqrt5}\right)$.

At $\left(-4/\sqrt5, -\frac{3}{\sqrt5}\right)$, $L_{xx} = 18 - \frac{9}{2}(3\sqrt5 + 5)$, $L_{xy} = 6$

$L_{yy} = 2 - 2(3\sqrt5 + 5)$, $dy = -3\,dx$. Then

$$d^2L = \left[-(8+6\sqrt5) - 36 - \frac{9}{2}(1+3\sqrt5)\right](dx)^2 < 0$$

which implies d^2L is negative definite and hence f has a maximum value at $\left(-\frac{4}{\sqrt5}, \frac{-3}{\sqrt5}\right)$

Thus the points of ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ which are at least distance and greatest distance from the given straight line $3x+y-9=0$ are respectively $\left(\frac{4}{\sqrt5}, \frac{3}{\sqrt5}\right)$ and $\left(-\frac{4}{\sqrt5}, \frac{-3}{\sqrt5}\right)$ and least distance is $\frac{9-3\sqrt5}{10}$ units and greatest distance is $\frac{9+3\sqrt5}{10}$ units.

4. Use the method of Lagrange's multipliers to show that the lengths of the semi-axes of the ellipse $ax^2 + 2bxy + cy^2 = 1$ are the square roots of the roots of $\begin{vmatrix} a\lambda - 1 & b\lambda \\ b\lambda & c\lambda - 1 \end{vmatrix} = 0$

Solution: Let r_1, r_2 be the lengths of the semi-axes of the ellipse $ax^2+2bxy+cy^2=1$. Then for some points (x,y) on the ellipse r_1^2 and r_2^2 are stationary values of $\lambda = x^2+y^2$, the square of the distance of (x,y) from the centre of the ellipse, i.e., origin.

We construct the Lagrangian function

$L(x,y) = x^2+y^2+\lambda'(ax^2+2bxy+cy^2-1)$ where the constant λ' is the Lagrange's multiplier

For stationary ~~vector~~ point (x,y) $dL=0$

$$\text{or,} \quad L_x = 2(x+\lambda'(ax+by)) = 0 \quad \cdots (1)$$

$$\text{and} \quad L_y = 2(y+\lambda'(cy+bx)) = 0 \quad \cdots (2)$$

Where (x,y) satisfies $ax^2+2bxy+cy^2=1 \quad \cdots (3)$

To determine λ', we use (1), (2) and (3) and obtain

$$\lambda + \lambda'(ax^2+2bxy+cy^2) = 0$$

$$\Rightarrow \lambda = -\lambda' \quad (\text{by } (3))$$

Then from (1) and (2), as x, y cannot both be zero.

$$\begin{vmatrix} 1+a\lambda' & b\lambda' \\ b\lambda' & 1+c\lambda' \end{vmatrix} = 0 \quad \text{i.e.,} \quad \begin{vmatrix} a\lambda-1 & b\lambda \\ b\lambda & c\lambda-1 \end{vmatrix} = 0$$

Which is a quadratic equation in λ giving two stationary value of λ and hence the square roots of the roots of $\begin{vmatrix} a\lambda-1 & b\lambda \\ b\lambda & c\lambda-1 \end{vmatrix} = 0$ gives the length of the semi-axes of the ellipse.

5. Use Lagrange's multiplier method to find the shortest distance between $(-1, 4)$ and the straight line $12x-5y+71=0$

Solution: Let (x, y) be any point on the given straight line. The square of its distance from $(-1, 4)$ is

$= (x+1)^2 + (y-4)^2$ whose minimum value is to be determined subject to the condition $12x-5y+71=0$

So, we construct Lagrangian function

$$L(x,y) = (x+1)^2 + (y-4)^2 + \lambda(12x-5y+71) \quad \cdots (1)$$

Where the constant λ is the Lagrangian multiplier.

From (1), $L_x = 2(x+1) + 12\lambda$, $L_y = 2(y-4) - 5\lambda$

For stationary point, $L_x = 0$, $L_y = 0$

$\Rightarrow \quad 2(x+1) + 12\lambda = 0$

and $\quad 2(y-4) - 5\lambda = 0$

$$\Rightarrow \quad \frac{\lambda}{2} = \frac{x+1}{-12} = \frac{y-4}{5}$$

Since (x, y) is a point on the straight line $12x-5y+71=0$

we have $12(-1-6\lambda) - 5\left(4+\frac{5}{2}\lambda\right) + 71 = 0$

$\Rightarrow \lambda = \frac{6}{13}$ and $x = -\frac{49}{13}$, $y = \frac{67}{13}$. Hence

$\left(-\frac{49}{13}, \frac{67}{13}\right)$ is the stationary point.

Now $dL = L_x dx + L_y dy$ and from $12x - 5y + 71 = 0$

$12dx - 5dy = 0$

$$d^2L = L_{xx}(dx)^2 + 2L_{xy}\,dx\,dy + L_{yy}(dy)^2$$

$$= 2\left[(dx)^2 + (dy)^2\right] > 0 \Rightarrow d^2L \text{ is positive definite.}$$

Therefore f has minimum value at $\left(-\frac{49}{13}, \frac{67}{13}\right)$ and the shortest distance $\sqrt{f_{min}} = 3$ units

THIS IS THE END OF MY PORTION OF NOTEE ON CC7