Theorem 1.2.9 Let $f: S \to \mathbb{R}^m$ and $g: S \to \mathbb{R}^m$ be two functions from S to $\mathbb{R}^m$, $S \subset \mathbb{R}^n$. Then ~~we have~~, if $\lim_{x \to a} f(x) = b$ and $\lim_{x \to a} g(x) = c$, we have

(i) $\lim_{x \to a} [f(x) + g(x)] = b + c$

(ii) $\lim_{x \to a} \lambda f(x) = \lambda b$, $\lambda \in \mathbb{R}$

(iii) $\lim_{x \to a} f(x) \cdot g(x) = b \cdot c$

(iv) $\lim_{x \to a} \| f(x) \| = \|b\|$

Proof: We prove only parts (iii) and (iv); proofs of (i) and (ii) are left as exercises

To prove (iii) we write

$$f(x) \cdot g(x) - b \cdot c = [f(x) - b] \cdot [g(x) - c] + b \cdot [g(x) - c] + c \cdot [f(x) - b]$$

Now we use the triangle inequality and Cauchy Schwarz inequality to obtain

$$0 \leq \| f(x) \cdot g(x) - b \cdot c \| \leq \|f(x) - b\| \|g(x) - c\| + \|b\| \|g(x) - c\| + \|c\| \|f(x) - b\|$$

Since $\|f(x) - b\| \to 0$ and $\|g(x) - c\| \to 0$ as $x \to a$, this shows that $\|f(x) \cdot g(x) - b \cdot c\| \to 0$ as $x \to a$, which proves (iii)

Taking $f(x) = g(x)$ in part (iii), we find that

$$\lim_{x \to a} \|f(x)\|^2 = \|b\|^2$$

from which we obtain (iv).

Example 1 Continuity and components of f where $f: S \to \mathbb{R}^m$ is a function from S to $\mathbb{R}^m$, $S \subset \mathbb{R}^n$. For $x \in S$, $f(x)$ has m components as $f(x) \in \mathbb{R}^m$

Let $f(x) = (f_1(x), f_2(x), \dots f_m(x))$ where $f_K : S \to \mathbb{R}$ are functions from S to $\mathbb{R}$, $K = 1, 2, \dots m$.

We shall prove that f is continuous at a point if and only if each component f_K is continuous at that point. Let e_K denote the Kth unit coordinate vector in $\mathbb{R}^m$ (i.e., all components of e_K are 0 except the Kth, which is equal to 1).

Then $f_K(x)$ is given by the dot product

$$f_K(x) = f(x) \cdot e_K$$

So, by part (iii) of Theorem 1.2.9 shows that each point of continuity of f is also a point of continuity of f_K. Moreover, since we have $f(x) = \sum_{K=1}^{m} f_K(x) e_K$, repeated application of parts (i) and (ii) of Theorem 1.2.9 shows that a point of continuity of all m components $f_1, f_2, \dots, f_m$ is also a point of continuity of f.

Example 2 Continuity of the identity function : The identity function $f : \mathbb{R}^n \to \mathbb{R}^n$, is given by $f(x) = x$, is continuous everywhere in $\mathbb{R}^n$. So, its components are also continuous everywhere in $\mathbb{R}^n$. The n components functions are given by, $f_K(x) = x_K$, $K = 1, 2, \dots, n$

Example 3. Continuity of linear transformations :

Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. We will prove that f is continuous at each point $a \in \mathbb{R}^n$.

By linearity, we have, $f(a+h) = f(a) + f(h)$

So, it suffices to prove that $f(h) \to 0$ as $h \to 0$

Writing h in terms of its components, we have

$h = (h_1, h_2, \dots, h_n) = \sum_{i=1}^{n} h_k e_k$ where e_k is the kth unit coordinate vector in $\mathbb{R}^n$ (i.e., all the components of e_k are 0 except the kth component which is equal to 1), $k = 1, 2, \dots, n$. Using linearity again, $f(h) = \sum_{k=1}^{n} h_k f(e_k)$. This shows that $f(h) \to 0$ as $h \to 0$

Example 4 Continuity of polynomials in n variables: A polynomial P is defined on $\mathbb{R}^n$ by a formula of the form

$$P(x) = \sum_{k_1=0}^{p_1} \cdots \sum_{k_n=0}^{p_n} c_{k_1 k_2 \dots k_n} x_1^{k_1} x_2^{k_2} \dots x_n^{k_n}, \quad x = (x_1, x_2, \dots, x_n)$$

$$c_{k_1 k_2 \dots k_n} \in \mathbb{R}$$

It is called a polynomial in n variables $x_1, x_2, \dots, x_n$.

It is continuous everywhere in $\mathbb{R}^n$ because it is a finite sum of products of scalar multiples of the function $f_1(x) = x_1, f_2(x_2) = x_2, \dots f_n(x) = x_n$.

For example, a polynomial in two variables x and y, given by $P(x,y) = \sum_{i=0}^{p} \sum_{j=0}^{q} c_{ij} x^i y^j$ is continuous at every point (x, y) in $\mathbb{R}^n$.

Example 5. Continuity of rational functions: The function $f: \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) = \frac{P(x)}{Q(x)}$, where P and Q

are polynomials in the components of $x = (x_1, x_2, \dots, x_n)$, is called a rational function. A rational function is continuous at each point where $Q(x) \neq 0$

Further examples of continuous function can be constructed with the help of the next theorem, which is concerned with the continuity of composite functions.

Theorem 1.2.10 Let $g: S \to \mathbb{R}^m$ and $f: T \to \mathbb{R}^k$ where $S \subset \mathbb{R}^n$ and $g(S) \subset T$, then the composite $f \circ g: S \to \mathbb{R}^k$ is defined as

$$f\circ g(x) = f(g(x)).$$

If g is continuous at a and if f is continuous at $g(a)$ then the composition function $f \circ g$ is continuous at a.

Proof: Let $y = g(x)$ and $b = g(a)$. Then we have $f(g(x)) - f(g(a)) = f(y) - f(b)$

By hypothesis $y \to b$ as $x \to a$, so we have

$$\lim_{\|x-a\| \to 0} \|f(g(x)) - f(g(a))\| = \lim_{\|y-b\| \to 0} \|f(y) - f(b)\| = 0$$

So,

$$\lim_{x \to a} f(g(x)) = f(g(a)).$$

So $f \circ g$ is continuous at a.

Example 6 The previous theorem implies the continuity of $h: \mathbb{R}^2 \to \mathbb{R}$ where $h(x,y)$ is given by formula such as $\sin(x^2y)$, $\log(x^2+y^2)$, $\log(\cos(x^2+y^2))$.