These examples are continuous at all points at which the functions are defined. The first is continuous at all points in the plane and the second is continuous at all points (x,y) except origin. The third is continuous ~~is~~ ~~continuous~~ at all points at which $x^2+y^2 \neq n\pi/2$, $n=1,3,5,\ldots$

This examples show that the discontinuities of a function of two variables may consist of isolated points, curves (no example given) or family of curves.

<u>Example 7</u> A function of two variables may be continuous in each variable separately and yet be discontinuous as a function of two variables together. This is illustrated by the following example:

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

For points on the x-axis we have $y=0$ and $f(x,y)=f(x,0)=0$, so the function has the constant value 0 everywhere on the x-axis. So, if we put $y=0$ and think of f as a function of x alone, f is continuous at $x=0$.

Similarly f has the constant value 0 at all points on the y-axis, so if we put $x=0$ and think of f as a function of y alone, f is continuous at $y=0$. However, as a function of two variables f is not continuous at $(0,0)$. In fact, at each point of the line $y = x$ (except at the origin) the function has the constant value $\frac{1}{2}$ because $f(x,x) = \frac{x^2}{2x^2} = \frac{1}{2}$; since there are points on this line arbitrarily close to $(0,0)$ and since $f(0,0) = 0 \neq \frac{1}{2}$, the function is not continuous at $(0,0)$.

~~Repeated~~ 1.2.11 Repeated limit or Iterated limit

Let N be a certain neighbourhood of (a,b) and $f(x,y)$ be defined on N. The limits

$$\lim_{x \to a}\left(\lim_{y \to b} f(x,y)\right) \text{ and } \lim_{y \to b}\left(\lim_{x \to a} f(x,y)\right)$$ are

called repeated or iterated limits.

In the first limit we fix x and find $\lim_{y \to b} f(x,y)$

if it exists, it will be a function of x alone; then we find $\lim_{x \to a}\left(\lim_{y \to b} f(x,y)\right)$. Similarly for the other limit we fix y and find $\lim_{x \to a} f(x,y)$. If it exists, it will be a function of y alone; then we find $\lim_{y \to b}\left(\lim_{x \to a} f(x,y)\right)$

Note: Existence of $\lim_{(x,y) \to (a,b)} f(x,y)$ (sometimes called double limit) will imply equality of repeated limits (if they exist) but not the converse. If, however, repeated limits are not equal, double limit can not exist.

Worked Examples 1. Find $\lim_{(x,y) \to (0,0)} f(x,y)$ where

$$f(x,y) = xy\frac{x^2 - y^2}{x^2 + y^2}$$

Solution: $|x| \leq \sqrt{x^2+y^2}$ and $|y| \leq \sqrt{x^2+y^2}$

Also $\left|\dfrac{x^2-y^2}{x^2+y^2}\right| \leq 1$. Let $\varepsilon > 0$ be given

So, $\left|f(x,y)-0\right| = \left|xy \cdot \dfrac{x^2-y^2}{x^2+y^2}\right| = |x||y|\left|\dfrac{x^2-y^2}{x^2+y^2}\right| \leq x^2+y^2 < \varepsilon$

if $0 < x^2+y^2 < \delta^2$ where $\delta = \sqrt{\varepsilon}$

So, by definition, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$

Theorem 1.2.12

Necessary condition for the existence of double limit.

Let $f: S \to \mathbb{R}$, $S \subseteq \mathbb{R}^2$. If $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$, then

$f(x, \phi(x)) \to L$ as $x \to a$, where ϕ is a real valued function of one variable such that $(x, \phi(x)) \in S$ for x belongs to the domain of ϕ and $\phi(x) \to b$ as $x \to a$

Proof: Let $\varepsilon > 0$ be given. Then $\exists\ \delta > 0$ such that

$|f(x,y) - L| < \varepsilon$ whenever $0 < (x-a)^2+(y-b)^2 < \delta^2$

or, $|f(x,y) - L| < \varepsilon$ whenever $0 < |x-a| < \delta$, $0 < |y-b| < \delta$ ... (1)

Given $\lim\limits_{x\to a} \phi(x) = b$, so corresponding to the above δ,

$\exists\ \eta > 0$ such that

$|\phi(x) - b| < \delta$ whenever $0 < |x-a| < \eta$ ... (2)

Let $\rho = \min\{\delta, \eta\}$

So, by (1) and (2), we have

$|f(x, \phi(x)) - L| < \varepsilon$ whenever $0 < |x-a| < \rho$

Consequently, $\lim\limits_{x\to a} f(x, \phi(x)) = L$

Corollary: If we can find out two functions $\phi_1(x)$ and $\phi_2(x)$ such that $\lim_{x\to a} f(x, \phi_1(x)) \neq \lim_{x\to a} f(x, \phi_2(x))$ where $(x, \phi_1(x))$ and $(x, \phi_2(x)) \in S$ for x belongs to the domain of ϕ_1 and ϕ_2, $\phi_1(x) \to b$, $\phi_2(x) \to b$ as $x \to a$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Worked Examples (continued)

2. Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

where $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y}, & x \neq y \\ 0, & x = y \end{cases}$

Solution: Let $(x,y)\to(0,0)$ along the path $x-y = mx^3$.

$$\frac{x^3+y^3}{x-y} = \frac{x^3 + (x-mx^3)^3}{mx^3} = \frac{1+(1-mx^2)^3}{m} \to \frac{2}{m} \text{ as } x\to 0$$

So, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

3. Show that $\lim_{(x,y)\to(0,0)} \dfrac{2xy}{\sqrt{x^2+y^2}} = 0$

Solution: Transferring to polar coordinates

$x = r\cos\theta$, $y = r\sin\theta$,

$$\frac{2xy}{\sqrt{x^2+y^2}} = r\sin 2\theta$$

Let $\varepsilon > 0$ be given

$|f(x,y) - 0| = |r\sin 2\theta| \leq |r| = \sqrt{x^2+y^2} < \varepsilon$

whenever $0 < x^2+y^2 < \delta^2$ where $\delta = \varepsilon$

So. $\lim_{(x,y)\to(0,0)} f(x,y) = 0$