Note: Sum, difference and modulus of continuous functions are continuous.

Worked Examples (Continued)

3. Let $f(x,y) = \begin{cases} (ax+by)\sin\frac{x}{y}, & y \neq 0 \\ 0, & y = 0 \end{cases}$ $(a, b \in \mathbb{R})$

Show that f is continuous at $(0,0)$

Solution: Let $\epsilon > 0$ be given

$$\left|f(x,y) - f(0,0)\right| = \left|(ax+by)\sin\frac{x}{y} - 0\right|$$

$$\leq |ax+by|$$

$$\leq |a||x| + |b||y|$$

$$\leq \frac{|a|\epsilon}{2(|a|+1)} + \frac{|b|\epsilon}{2(|b|+1)}$$

whenever $|x-0| < \delta_1 = \frac{\epsilon}{2(|a|+1)}$, $|y-0| < \delta_2 = \frac{\epsilon}{2(|b|+1)}$

If $\delta = \min\{\delta_1, \delta_2\}$, we have

$$|f(x,y) - f(0,0)| < \epsilon \text{ whenever } |x-0| < \delta \text{ and } |y-0| < \delta$$

So f is continuous at $(0,0)$

4. Let $f(x,y) = \begin{cases} xy\log(x^2+y^2), & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$

Prove that f is continuous at $(0,0)$

Solution: For $0 < (x^2+y^2) < 1$, $\log(x^2+y^2) < 0$

Thus for $0 < (x^2+y^2) < 1$, we have

$$|f(x,y) - f(0,0)| = -|xy|\log(x^2+y^2) \leq -\frac{1}{2}(x^2+y^2)\log(x^2+y^2)$$

(as AM of two positive quantities $\geq$ their GM)

If we take $t = x^2+y^2$, by L'Hospital's rule

$$\lim_{t\to 0+} t\log t = 0$$

So, given $\epsilon>0$, $\exists\ \delta>0$ such that

$$|t\log t - 0| < 2\epsilon \quad \text{whenever } 0<t<\delta$$

$$\Rightarrow \left|(x^2+y^2)\log(x^2+y^2)\right| < 2\epsilon \quad \text{whenever } 0<t<\delta$$

For $\eta = \min\{1, \delta\}$, we have for $0<x^2+y^2<\eta$,

$|f(x,y) - f(0,0)| < \epsilon \Rightarrow f$ is continuous at $(0,0)$.

5. Show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. But the repeated limits do not exist where $f(x,y)$ is defined as

$$f(x,y) = \begin{cases} x\sin\frac{1}{y} + y\sin\frac{1}{x}, & xy\neq 0 \\ 0, & xy=0 \end{cases}$$

Solution: ~~Here $|f(x,y)-0|$~~ Let $\epsilon>0$ be given.

Now $|f(x,y) - 0| = \left|x\sin\frac{1}{y} + y\sin\frac{1}{x}\right|$

$$\leq |x|\left|\sin\frac{1}{y}\right| + |y|\left|\sin\frac{1}{x}\right|$$

$$\leq |x| + |y| \quad (\because |\sin x|\leq 1)$$

$$< \epsilon$$

if $|x-0|<\delta$ and $|y-0|<\delta$

~~we choose~~ when $\delta = \frac{\epsilon}{2}$

So, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

Now as $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist

So, $\lim_{x\to 0}\left(x\sin\frac{1}{y}+y\sin\frac{1}{x}\right)$ does not exist when y is fixed

and also $\lim_{y\to 0}\left(x\sin\frac{1}{y}+y\sin\frac{1}{x}\right)$ does not exist when x is fixed. So the repeated limits

$\lim_{x\to 0}\lim_{y\to 0} f(x,y)$ and $\lim_{y\to 0}\lim_{x\to 0} f(x,y)$ do not exist.

6. If $f(x,y)=\begin{cases}1 & \text{when } xy\neq 0\\ 0 & \text{when } xy=0\end{cases}$

Show that $\lim_{x\to 0}\lim_{y\to 0} f(x,y)=\lim_{y\to 0}\lim_{x\to 0} f(x,y)$ but

$\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Solution: Let us keep $x=a\neq 0$ fixed Then $f(x,y)=f(a,y)$

So, $\lim_{y\to 0} f(x,y)=\lim_{y\to 0} f(a,y)=\lim_{y\to 0} 1=1$

So, $\lim_{x\to 0}\lim_{y\to 0} f(x,y)=\lim_{x\to 0} 1=1$

Again if we keep $y=b\neq 0$ fixed then $f(x,y)=f(x,b)$

and $\lim_{x\to 0} f(x,y)=\lim_{x\to 0} f(x,b)=\lim_{x\to 0} 1=1$

So, $\lim_{y\to 0}\lim_{x\to 0} f(x,y)=\lim_{y\to 0} 1=1$

So, $\lim_{x\to 0}\lim_{y\to 0} f(x,y)=\lim_{y\to 0}\lim_{x\to 0} f(x,y)=1$

Now along the x-axis, i.e., for $y=0$

$f(x,y)=f(x,0)=0$ and along $y=x$,

$f(x,y)=f(x,x)=1$ So, if $(x,y)\to(0,0)$

along these two directions

limits are different. When $(x,y)\to(0,0)$ through x axis

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$ and when $(x,y)\to(0,0)$ through $y=x$

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = 1$, So, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

7. Let $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y} & \text{when } x\neq y \\ 0 & \text{when } x=y \end{cases}$

Show that f is not continuous at $(0,0)$.

Solution: Let $(x,y)\to(0,0)$ along $x-y=mx^3$

So, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{x\to 0} \dfrac{x^3+(x-mx^3)^3}{mx^3}$

$= \lim\limits_{x\to 0} \dfrac{1+(1-mx^2)^3}{m} = \dfrac{2}{m}$ which are

different for different values of m.

So, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist. Hence f is

not continuous at $(0,0)$

Exercises 1. Let $f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

Show that f is continuous at $(0,0)$

2. Show that $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$

is not continuous at $(0,0)$.