1.3. The derivative of a function $f: S \to \mathbb{R}$, where $S \subseteq \mathbb{R}^n$

Let a be an interior point of S. We wish to study how the function changes from a to a nearby point. For example, suppose $f(a)$ is the temperature at a point a in a heated room with an open window. If we move towards the window, the temperature will decrease, but if we move towards the heater it will increase. In general, the manner in which f changes will depend upon the direction in which we move away from a.

Suppose we specify the direction by some $y \in \mathbb{R}^n$. That is, suppose we move from a toward another point $a+y$ along the line segment joining a and $a+y$. Each point on this segment has the form $a+hy$ where $h \in \mathbb{R}$. An example is shown in Figure 1. The distance from a to $a+hy$ is $\|hy\| = |h|\|y\|$. Since a is an interior point of S, there is an n-ball $B(a;r) = \{x \in \mathbb{R}^n : \|x-a\| < r\}$ lying entirely in S. If h is chosen so that $|h|\|y\| < r$, the segment from a to $a+hy$ will lie in S (See Figure 2).

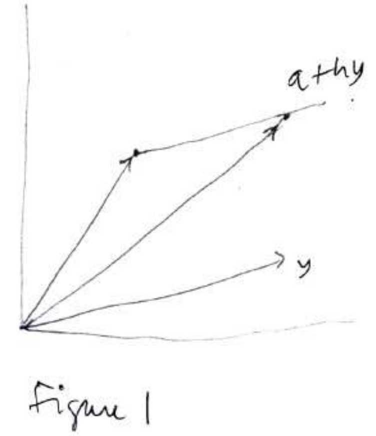

Figure 1

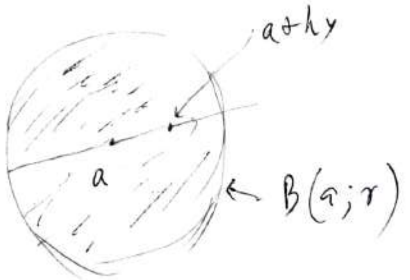

Figure 2

We keep $h \neq 0$ but small enough to guarantee that $a+hy \in S$ and we form the difference quotient

$$\frac{f(a+hy)-f(a)}{h} \quad \cdots \quad (1)$$

The numerator of this quotient tells us how much the function changes when we move from a to $a+hy$. The quotient itself is called the average rate of change of f over the line segment joining a to $a+hy$. We are interested in the behaviour of the quotient as $h \to 0$. This leads to the following definition:

Definition of the derivative of a function $f: S \to \mathbb{R}$ where $S \subset \mathbb{R}^n$ with respect to a point $y \in \mathbb{R}^n$:

Given a function $f: S \to \mathbb{R}$, where $S \subseteq \mathbb{R}^n$. Let a be an interior point of S and let y an arbitrary point in $\mathbb{R}^n$. The derivative of f at a with respect to y is denoted by the symbol $f'(a;y)$ and is defined by the equation

$$f'(a;y) = \lim_{h \to 0} \frac{f(a+hy)-f(a)}{h} \quad \cdots \quad (2)$$

when the limit on the right hand side exists.

Example 1 If $y = O = (\overbrace{0,0,\cdots 0}^{n \text{ times}})$, the difference quotient (1) is 0 for ever $h \neq 0$, so $f'(a;O)$ always exists and equals 0.

Example 2 Derivative of a linear transformation: If $f: S \to \mathbb{R}$ be linear, then $f(a+hy) = f(a) + hf(y)$ and the difference quotient (1) is equal to $f(y)$ for every $h \neq 0$. In this case $f'(a;y)$ always exists and is given by $f'(a;y) = f(y)$

for every a in S and every y in $\mathbb{R}^n$. In other words, the derivative of a linear transformation with respect to y is equal to the value of the function at y.

To study how f behaves on the line passing through a and $a+y$ for $y \neq 0$, we introduce the function
$$g(t) = f(a+ty)$$

The next theorem relates the derivatives $g'(t)$ and $f'(a+ty;y)$.

Theorem 1-3.1 Let $g(t) = f(a+ty)$. If one of the derivatives $g'(t)$ or $f(a+ty;y)$ exists then the other also exists and the two are equal,
$$g'(t) = f'(a+ty;y) \quad \cdots \quad (3)$$

In particular, when $t=0$ we have $g'(0) = f'(a;y)$

Proof: Forming the difference quotient for g, we have
$$\frac{g(t+h) - g(t)}{h} = \frac{f(a+ty+hy) - f(a+ty)}{h}$$

Letting $h \to 0$, we obtain (3)

Example 3 Compute $f'(a;y)$ if $f(x) = \|x\|^2$, $x \in \mathbb{R}^n$

Solution: We let $g(t) = f(a+ty) = (a+ty)\cdot(a+ty)$

$= a.a + 2t\,a.y + t^2 y.y$

So, $g'(t) = 2a.y + 2t\,y.y$, so $g'(0) = f'(a;y) = 2a.y$

<u>Corollary of Theorem 1.3.1</u> (Mean value theorem for the function $f: S \to \mathbb{R}$ where $S \subseteq \mathbb{R}^n$). Assume the derivative $f'(a+ty; y)$ exists for each t in the interval $0 \le t \le 1$. Then for some real θ in the open interval $0 < \theta < 1$, we have

$$f(a+y) - f(a) = f'(z; y) \quad \text{where } z = a + \theta y$$

Proof: Let $g(t) = f(a+ty)$. Applying the one dimensional mean value theorem to g on the interval $[0,1]$, we have $g(1) - g(0) = g'(\theta)$, where $0 < \theta < 1$

Since $g(1) - g(0) = f(a+y) - f(a)$ and $g'(\theta) = f'(a+\theta y; y)$

this completes the proof.

1.4. <u>Directional derivatives and partial derivatives</u>

In the special case when y is a unit vector, that is, when $\|y\| = 1$, the distance between a and $a+hy$ is $\|a+hy - a\| = |h|\,\|y\| = |h|$. In this case the difference quotient (1) of Page-22 represents the average rate of change of f per unit distance along the segment joining a to $a+hy$; the derivative $f'(a;y)$ is called a directional derivative.

Definition of Directional and partial derivatives: If y