is a unit vector, the derivative $f'(a;y)$ is called the directional derivative of f at a in the direction of y. In particular, if $y = e_k$ (the kth unit coordinate vector $(0, 0, \cdots \overset{k\text{th component}}{1}, 0, \cdots 0)$), the directional derivative $f'(a; e_k)$ is called the partial derivative with respect to e_k and is also denoted by $D_k f(a)$. Thus

$$D_k f(a) = f'(a; e_k).$$

The following notations are also used for the partial derivative $D_k f(a)$:

$D_k f(a_1, a_2, \cdots, a_n)$, $\dfrac{\partial f}{\partial x_k}(a_1, a_2, \cdots, a_n)$ and $f_{x_k}(a_1, a_2, \cdots, a_n)$

In $\mathbb{R}^2$, the unit coordinate vectors are denoted by $\hat{i}$ and $\hat{j}$. If $a = (a, b)$, the partial derivative $f'(a; \hat{i})$ and $f'(a; \hat{j})$ are also written as

$$\frac{\partial f}{\partial x}(a,b) \text{ and } \frac{\partial f}{\partial y}(a,b) \text{ respectively.}$$

In $\mathbb{R}^3$, if $a = (a, b, c)$ the partial derivative $D_1 f(a)$, $D_2 f(a)$ and $D_3 f(a)$ are also denoted by

$$\frac{\partial f}{\partial x}(a,b,c), \quad \frac{\partial f}{\partial y}(a,b,c) \text{ and } \frac{\partial f}{\partial z}(a,b,c)$$

1.5. Partial differentiation produces new functions $D_1 f, D_2 f, \cdots, D_n f$ from $\mathbb{R}^n$ to $\mathbb{R}$ from the function f. The partial derivatives of $D_1 f, D_2 f, \ldots, D_n f$ are called second-order partial derivatives of f.

For function of two variables there are four second-order partial derivatives of f, which are written as follows:

$D_1(D_1 f) = \dfrac{\partial^2 f}{\partial x^2}$, $D_1(D_2 f) = \dfrac{\partial^2 f}{\partial x \partial y}$, $D_2(D_1 f) = \dfrac{\partial^2 f}{\partial y \partial x}$

$D_2(D_2 f) = \dfrac{\partial^2 f}{\partial y^2}$. Note that $D_1(D_2 f)$ means the partial derivative of $D_2 f$ with respect to the first variable. We sometimes use the notation $D_{i,j} f$ for the second-order partial derivative $D_i(D_j f)$. For example, $D_{1,2} f = D_1(D_2 f)$.

In the notation we indicate the order of the derivative by writing $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$.

This may or may not be equal to the other mixed partial derivative,

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Note: $f_x = \dfrac{\partial f}{\partial x}$, $f_y = \dfrac{\partial f}{\partial y}$, $f_{xx} = \dfrac{\partial^2 f}{\partial x^2}$, $f_{yy} = \dfrac{\partial^2 f}{\partial y^2}$,

$f_{xy} = \dfrac{\partial^2 f}{\partial x \partial y}$ and $f_{yx} = \dfrac{\partial^2 f}{\partial y \partial x}$ are also used as notations.

Worked Examples :

1. If $u = (1-2xy+y^2)^{-\frac{1}{2}}$, show that $\dfrac{\partial}{\partial x}\left\{(1-x^2)\dfrac{\partial u}{\partial x}\right\} + \dfrac{\partial}{\partial y}\left(y^2 \dfrac{\partial u}{\partial y}\right) = 0$

Solution: Here $u_x = y(1-2xy+y^2)^{-3/2} = yu^3$

and $u_y = (x-y)(1-2xy+y^2)^{-3/2} = (x-y)u^3$

$\dfrac{\partial}{\partial x}(u_x) = y \cdot 3u^2 \cdot u_x = 3y^2 u^5$

and $\frac{\partial}{\partial y}(u_y) = -u^3 + (x-y)\, 3u^2 \cdot u_y = -u^3 + 3(x-y)^2 u^5$

Given expression $= -2x u_x + 2y \cdot u_y + (1-x^2) \cdot 3y^2 u^5 + y^2 \{-u^3 + 3u^5 (x-y)^2\}$

$$= -2xy u^3 + 2y(x-y) u^3 + (1-x^2)\, 3y^2 u^5 - y^2 u^3 + 3y^2 u^5 (x-y)^2$$

$$= -3y^2 u^3 + 3y^2 u^5 - x^2 \cdot 3y^2 u^5 + 3y^2 u^5 (x^2 - 2xy + y^2)$$

$$= -3y^2 u^3 + 3y^2 u^5 - x^2 \cdot 3y^2 u^5 + x^2 \cdot 3y^2 u^5 - 2xy \cdot 3y^2 u^5 + 3y^4 u^5$$

$$= -3y^2 u^3 + 3y^2 u^5 (1 - 2xy + y^2)$$

$$= -3y^2 u^3 + 3y^2 u^5 \cdot u^{-2}$$

$$= -3y^2 u^3 + 3y^2 u^3 = 0$$

2. Let $f(x,y) = \begin{cases} x \sin \frac{1}{x} + y \sin \frac{1}{y}, & xy \neq 0 \\ x \sin \frac{1}{x}, & x \neq 0,\ y = 0 \\ y \sin \frac{1}{y}, & x = 0,\ y \neq 0 \\ 0, & x = 0,\ y = 0 \end{cases}$

Show that f is continuous at $(0,0)$ but f_x and f_y at $(0,0)$ do not exist.

Solution: Here f is continuous at $(0,0)$ (Show it)

As $\lim_{t \to 0} \sin \frac{1}{t}$ does not exist, so neither f_x nor f_y exists at $(0,0)$

3. Let $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$

Show that $f_x(0,0) = f_y(0,0) = 0$ but f is discontinuous at $(0,0)$

Solution: $\lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0 \Rightarrow f_x(0,0) = 0$

Similarly, $f_y(0,0) = \lim_{k\to 0} \frac{f(0, 0+k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$

$\Rightarrow f_y(0,0) = 0$

Let $(x,y) \to (0,0)$ along $y = mx$

Then $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{mx^2}{x^2+m^2x^2} = \frac{m}{1+m^2}$

Which is different for different m.

So, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

So, f is discontinuous at $(0,0)$

3. Let $f(x,y) = \begin{cases} x^2 \sin\frac{1}{x} + y^2 \sin\frac{1}{y} & \text{when } x \neq 0, y \neq 0 \\ x^2 \sin\frac{1}{x} & \text{when } y = 0, x \neq 0 \\ y^2 \sin\frac{1}{y} & \text{when } x = 0, y \neq 0 \\ 0 & \text{when } x = 0, y = 0 \end{cases}$

Find $f_x(0,y)$ and $f_y(x,0)$

Solution: For any fixed y, $\lim_{x\to 0} \frac{f(x,y) - f(0,y)}{x}$

$= \lim_{x\to 0} \frac{x^2 \sin\frac{1}{x} + y^2 \sin\frac{1}{y} - y^2 \sin\frac{1}{y}}{x} = \lim_{x\to 0} \frac{x^2 \sin\frac{1}{x}}{x}$

$= \lim_{x\to 0} x \sin\frac{1}{x} = 0$

Also, for any fixed x, $\lim_{y\to 0} \frac{f(x,y) - f(x,0)}{y}$

$= \lim_{y\to 0} \frac{x^2 \sin\frac{1}{x} + y^2 \sin\frac{1}{y} - x^2 \sin\frac{1}{x}}{y} = \lim_{y\to 0} \frac{y^2 \sin\frac{1}{y}}{y}$

$= \lim_{y\to 0} y \sin\frac{1}{y} = 0$

So, $f_x(0,y) = 0$ and $f_y(x,0) = 0$