1.7 Directional derivatives and continuity :

In the one dimensional theory, existence of the derivative of a function f at a point implies the continuity at that point. This is easily proved by choosing an $h \neq 0$ and writing

$$f(a+h) - f(a) = \frac{f(a+h) - f(a)}{h} \cdot h$$

As $h \to 0$, the right side tends to the limit $f'(a) \cdot 0 = 0$ and hence $f(a+h) \to f(a)$. This shows that the existence of $f'(a)$ implies continuity of f at a

Suppose we apply the same argument to a function $f : S \to \mathbb{R}$ where $S \subseteq \mathbb{R}^n (n>1)$. Assume the derivative $f'(a;y)$ exists for some y. Then if $h \neq 0$, we can write

$$f(a+hy) - f(a) = \frac{f(a+hy) - f(a)}{h} \cdot h$$

As $h \to 0$, the right side tends to the limit $f'(a;y) \cdot 0 = 0$

Hence the existence of $f'(a;y)$ for a given y implies that

$$\lim_{h \to 0} f(a+hy) = f(a) \quad \text{for the same } y.$$

This means that $f(x) \to f(a)$ as $x \to a$ along a straight line through a having the direction y. If $f'(a;y)$ exists for every vector y, then $f(x) \to f(a)$ as $x \to a$ along every line through a. This seems to suggest that f is continuous at a. Surprisingly enough, this conclusion need not be true. The next example describes an $f : S \to \mathbb{R}$, $S \subseteq \mathbb{R}^n$ which has a ~~derivative~~ directional derivative in every direct at 0 but which is not

Continuous at 0.

Example: Let us consider the function $f:\mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Let $a=(0,0)$ and let $y=(c,d)$ be any vector. If $c \neq 0$ and $h \neq 0$, we have

$$\frac{f(a+hy)-f(a)}{h} = \frac{f(hy)-f(a)}{h}$$

$$= \frac{f(hc,hd)-f(0,0)}{h} = \frac{hc.h^2d^2}{h(h^2c^2+h^4d^4)} = \frac{cd^2}{c^2+h^2d^4}$$

Letting $h \to 0$, we find $f'(a;y) = \dfrac{cd^2}{c^2} = \dfrac{d^2}{c}$

If $y=(0,d)$ we find in a similar way, that $f'(a;y)=0$. So, $f'(a;y)$ exists for all direction y where $a=(0,0)$. Also $f(x) \to 0$ as $x \to (0,0)$ along any straight line through origin. However, at each point of the parabola ~~(except at the origin), the function f has~~ $x=y^2$ (except at the origin) the function f has the value $\dfrac{y^4}{y^4+y^4} = \dfrac{1}{2}$. Since such points exist arbitrarily close to the origin and since $f(0,0)=0$, the function f is not continuous at $(0,0)$.

The previous example shows that the existence of all directional derivatives at a point fails to imply continuity at a point.

~~For this reason, directional derivatives at a point fail to imply continuity at that point~~

For this reason, directional derivatives are a somewhat unsatisfactory extension of the one-dimensional concept of derivative. A more suitable generalization exists which implies continuity and at the same time, permits us to extend the principal theorems of one-dimensional derivative theory to the higher dimensional cases. This is called the total derivative.

1.8 The Total derivative:

We recall that in the one-dimensional case, a function f with a derivative at a can be approximated near a by a linear Taylor polynomial. If $f'(a)$ exists we let $E(a,h)$ denote the difference

$$E(a,h) = \frac{f(a+h)-f(a)}{h} - f'(a) \quad \text{if } h \neq 0 \quad \text{--(1)}$$

and we define $E(a,0) = 0$. From (1) we obtain the formula $f(a+h) = f(a) + f'(a)h + hE(a,h)$,

an equation which holds also for $h=0$. This is the first-order Taylor formula for approximating $f(a+h)-f(a)$ by $f'(a)h$. The error committed is $hE(a,h)$. From (1) we see that $E(a,h) \to 0$ as $h \to 0$. Therefore the error $hE(a,h)$ is of smaller order than h for small h. This property of approximating a differentiable function by a linear function suggests a way of extending the concept of differentiability to the higher dimensional cases.

Let $f : S \to \mathbb{R}$ be a function, $S \subseteq \mathbb{R}^n$ $(n > 1)$. Let a be an interior point of S, and let $B(a; r) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$ be an n-ball lying in S. Let $v \in \mathbb{R}^n$ such that $\|v\| < r$, so that

$a + v \in B(a; r)$

We say that f is differentiable at a if $\exists$ a linear transformation $T_a : \mathbb{R}^n \to \mathbb{R}$ from $\mathbb{R}^n$ to $\mathbb{R}$ and a function $E(a, v) : \mathbb{R}^n \to \mathbb{R}$ such that

$$f(a+v) = f(a) + T_a(v) + \|v\| \, E(a, v), \quad \cdots \quad (2)$$

for $\|v\| < r$, where $E(a, v) \to 0$ as $\|v\| \to 0$.

The linear transformation T_a is called the total derivative of f at a

Note: The total derivative T_a is a linear transformation, not a number. The function value $T_a(v)$ is a real number; it is defined for every $v \in \mathbb{R}^n$. The total derivative was introduced by W. H. Young in 1908 and by M. Frechet in 1911 in a more general context.

Equation (2) which holds for $\|v\| < r$, is called a first-order Taylor formula for $f(a+v)$. It gives a linear approximation, $T_a(v)$ to the difference $f(a+v) - f(a)$. The error in the approximation is $\|v\| \, E(a, v)$, a term which is of smaller order than $\|v\|$ as $\|v\| \to 0$; that is,

$$E(a, v) = o(\|v\|) \text{ as } \|v\| \to 0.$$

The next theorem shows that if the total derivative