2nd Part

proof, Let $\varepsilon > 0$, Since $f \in \mathcal{R}[a,b]$, $\exists$ a partition P_1 of $[a,b]$ such that $U(P_1, f) < \int_a^b f(x)dx + \frac{\varepsilon}{2}$

Since $g \in \mathcal{R}[a,b]$, $\exists$ a partition P_2 of $[a,b]$ such that $U(P_2, g) < \int_a^b g(x)dx + \varepsilon/2$

Let $P_0 = P_1 \cup P_2$. So, P_0 is a refinement of both P_1 and P_2 and $U(P_0, f) \le U(P_1, f) < \int_a^b f(x)dx + \varepsilon/2$

and $U(P_0, g) \le U(P_2, g) < \int_a^b g(x)dx + \varepsilon/2$

So, $U(P_0, f+g) \le U(P_0, f) + U(P_0, g) < \int_a^b f(x)dx + \int_a^b g(x)dx + \varepsilon$

Since $(f+g)$ is integrable on $[a,b]$, $\int_a^b (f(x)+g(x))dx \le U(P_0, f+g)$

Hence $\int_a^b (f(x)+g(x))dx < \int_a^b f(x)dx + \int_a^b g(x)dx + \varepsilon \quad \ldots (i)$

Considering the lower sums, by similar arguments, we have $\int_a^b (f(x)+g(x))\,dx > \int_a^b f(x)dx + \int_a^b f(x)dx - \varepsilon$

From (i) and (ii), we have $\left| \int_a^b (f(x)+g(x))dx - \int_a^b f(x)dx - \int_a^b g(x)dx \right| < \varepsilon$

This holds for every positive ε.

So, $\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

Note: For a finite number of functions $f_1, f_2, \ldots, f_n$ each integrable on $[a,b]$, $f_1 + f_2 + \cdots + f_n$ is integrable on $[a,b]$ and $\int_a^b (f_1(x)+f_2(x)+\cdots+f_n(x))dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx + \cdots + \int_a^b f_n(x)dx$

Theorem 1.7.2 Let $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ and $c \in \mathbb{R}$. Then cf is integrable on $[a,b]$ and $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

Proof: Since $f \in \mathcal{R}[a,b]$, f is bounded on $[a,b]$. So, cf is bounded on $[a,b]$

Case 1 $c = 0$ $cf(x) = 0 \ \forall x \in [a,b]$. cf is integrable on $[a,b]$ and $\int_a^b cf(x)dx = 0$ and so $\int cf(x)dx = c\int f(x)dx$

Case 2 $c > 0$

$\underline{\int_a^b} f(x)dx$ = the supremum of the set $\{L(P,f) : P \in \mathcal{P}[a,b]\}$

$\overline{\int_a^b} f(x)dx$ = the infimum of the set $\{U(P,f) : P \in \mathcal{P}[a,b]\}$

$\underline{\int_a^b} cf(x)dx$ = the supremum of the set $\{L(P,cf) : P \in \mathcal{P}[a,b]\}$

= the supremum of the set $\{c\,L(P,f) : P \in \mathcal{P}[a,b]\}$

= c. the supremum of the set $\{L(P,f) : P \in \mathcal{P}[a,b]\}$

$= c \cdot \underline{\int_a^b} f(x)dx$

Similarly $\overline{\int_a^b} cf(x)dx = c.\overline{\int_a^b} f(x)dx$

Since $f \in \mathcal{R}[a,b]$, $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx = \int_a^b f(x)dx$

Hence $\underline{\int_a^b} cf(x)dx = \overline{\int_a^b} cf(x)dx = c\int_a^b f(x)dx$

This shows that cf is integrable on $[a,b]$ and $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

Case 3 $c < 0$ Similar proof.

Note : The two theorems Theorem 1.7.1 and Theorem 1.7.2 together establish that Riemann integrals satisfy linearity property.

Theorem 1.7.3 Let a function $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. Then $|f|$ is integrable on $[a,b]$

Proof: Since $f \in \mathcal{R}[a,b]$, f is bounded on $[a,b]$. So, $\exists$ a positive real number k such that $|f(x)| < k$ for all $x \in [a,b]$

Now $\left| |f|(x) \right| = |f|(x) = |f(x)| < k \quad \forall x \in [a,b]$.

This shows that $|f|$ is bounded on $[a,b]$

Let us choose $\varepsilon > 0$. Since f is integrable on $[a,b]$, $\exists$ a partition P of $[a,b]$ such that

$U(P,f) - L(P,f) < \varepsilon$

Let $P = (x_0, x_1, \ldots, x_n)$, where $a = x_0 < x_1 < \cdots < x_n = b$

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$

$$M_r' = \sup_{x \in [x_{r-1}, x_r]} |f|(x) \qquad m_r' = \inf_{x \in [x_{r-1}, x_r]} |f|(x),$$

for $r = 1, 2, \ldots, n$

For any two points α, β in $[x_{r-1}, x_r]$, we have

$$\left| |f|(\alpha) - |f|(\beta) \right| = \left| |f(\alpha)| - |f(\beta)| \right| \le |f(\alpha) - f(\beta)| \cdots (i)$$

We use here an important property of a bounded subset of $\mathbb{R}$. If S be a non-empty bounded subset of $\mathbb{R}$ with $\sup S = M$ and $\inf S = m$, then the supremum of the set $\{|x-y| : x \in S, y \in S\}$ is $M-m$

Since f is bounded in $[x_{r-1}, x_r]$ with $\sup_{x \in [x_{r-1}, x_r]} f(x) = M_r$ and $\inf_{x \in [x_{r-1}, x_r]} f(x) = m_r$, the supremum of the set $\{|f(\alpha) - f(\beta)| : \alpha, \beta \in [x_{r-1}, x_r]\}$ is $M_r - m_r$

Since $|f|$ is bounded on $[x_{r-1}, x_r]$ with $\sup_{x \in [x_{r-1}, x_r]} |f|(x) = M_r'$ and $\inf_{x \in [x_{r-1}, x_r]} |f|(x) = m_r'$, the supremum of the set $\{||f|(\alpha) - |f|(\beta)| : \alpha, \beta \in [x_{r-1}, x_r]\}$ is $M_r' - m_r'$

From the inequality (i) it follows that $M_r - m_r$ is an upper bound of the set $\{|f|(\alpha) - |f|(\beta) : \alpha, \beta \in [x_{r-1}, x_r]\}$

So, $M_r' - m_r' \le M_r - m_r$. This holds for $r = 1, 2, \ldots, n$

So, $U(P, |f|) - L(P, |f|)$

$$= (M_1' - m_1')(x_1 - x_0) + \cdots + (M_n' - m_n')(x_n - x_{n-1})$$

$$\le (M_1 - m_1)(x_1 - x_0) + \cdots + (M_n - m_n)(x_n - x_{n-1})$$

$$= U(P, f) - L(P, f) < \varepsilon$$

This being a sufficient condition for integrability, $|f|$ is integrable on $[a,b]$. This completes the proof.

Note: The converse of the theorem is not true. For example, let $f : [a,b] \to \mathbb{R}$ be defined by

$$f(x) = 1, \ x \in [a,b] \cap \mathbb{Q}$$
$$= -1, \ x \in [a,b] - \mathbb{Q}$$

Then f is not integrable on $[a,b]$. But $|f|(x) = 1$ for all $x \in [a,b]$ and $|f|$ is integrable on $[a,b]$.