Theorem 1.7.9 Let a function $f:[a,b]\to\mathbb{R}$ be integrable on $[a,b]$. Then f^2 is integrable on $[a,b]$

Proof: Since $f\in\mathcal{R}[a,b]$, f is bounded on $[a,b]$.

So, $\exists$ a positive real number k such that $|f(x)|\le k$ for all $x\in[a,b]$. So, $|f^2(x)|\le k^2$ for all $x\in[a,b]$

This shows that f^2 is bounded on $[a,b]$.

Let us choose $\varepsilon>0$.

Since f is integrable on $[a,b]$, $\exists$ a partition P of $[a,b]$ such that $U(P,f)-L(P,f)<\frac{\varepsilon}{2k}$.

Let $P=(x_0,x_1,\dots,x_n)$, where $a=x_0<x_1<\dots<x_n=b$

Let $M_r=\sup\limits_{x\in[x_{r-1},x_r]} f(x)$, $m_r=\inf\limits_{x\in[x_{r-1},x_r]} f(x)$

and $M_r'=\sup\limits_{x\in[x_{r-1},x_r]} f^2(x)$ & $m_r'=\inf\limits_{x\in[x_{r-1},x_r]} f^2(x)$ for $r=1,2,\dots,n$

For any two points α,β in $[x_{r-1},x_r]$, we have

$$
\begin{aligned}
|f^2(\alpha)-f^2(\beta)| &= |\{f(\alpha)\}^2-\{f(\beta)\}^2| \\
&= |f(\alpha)+f(\beta)|\,|f(\alpha)-f(\beta)| \\
&\le 2k\,|f(\alpha)-f(\beta)| \qquad \dots \text{(i)}
\end{aligned}
$$

Since f is bounded on $[x_{r-1},x_r]$ with $\sup\limits_{x\in[x_{r-1},x_r]} f(x)=M_r$ and $\inf\limits_{x\in[x_{r-1},x_r]} f(x)=m_r$, the supremum of the set $\{|f(\alpha)-f(\beta)|:\alpha,\beta\in[x_{r-1},x_r]\}$ is M_r-m_r. Since f^2 is bounded on $[x_{r-1},x_r]$

with $\sup\limits_{x \in [x_{r-1}, x_r]} f^2(x) = M_r'$ and $\inf\limits_{x \in [x_{r-1}, x_r]} f^2(x) = m_r'$, the supremum of the set $\left\{ |f^2(x) - f^2(p)| : x, p \in [x_{r-1}, x_r] \right\} = M_r' - m_r'$

It follows from the inequality (*) that $2k(M_r - m_r)$ is an upper bound of the set $\left\{ |f^2(x) - f^2(p)| : x, p \in [x_{r-1}, x_r] \right\}$

So, $M_r' - m_r' \leq 2k(M_r - m_r)$. This holds for $r = 1, 2, \ldots, n$

So,
$$U(P, f^2) - L(P, f^2) = \sum_{r=1}^{n} (M_r' - m_r')(x_r - x_{r-1})$$
$$\leq 2k \sum_{r=1}^{n} (M_r - m_r)(x_r - x_{r-1})$$
$$= 2k\left(U(P, f) - L(P, f)\right) < \epsilon$$

This being a sufficient condition for integrability, f^2 is integrable on $[a, b]$. This completes the proof.

NOTE: For supremum result used in the above theorem, see the result of worked example 5, page 32 of Real Analysis - S. K. Mapa.

Theorem 1.7.5 Let the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be both integrable on $[a, b]$. Then fg is is integrable on $[a, b]$.

Proof: Since $f \in \mathcal{R}[a, b]$, f is bounded on $[a, b]$.
Since $g \in \mathcal{R}[a, b]$, g is bounded on $[a, b]$.
Therefore fg is bounded on $[a, b]$
Now $fg = \frac{1}{2}\left((f+g)^2 - \frac{1}{2}f^2 - \frac{1}{2}g^2 \right)$
Since $f \in \mathcal{R}[a, b]$ and $g \in \mathcal{R}[a, b]$, $\frac{1}{2}(f+g)^2$, $\frac{1}{2}f^2$, $\frac{1}{2}g^2$ are all integrable on $[a, b]$ by Theorems 1.7.1, 1.7.2 and 1.7.3.

Hence fg is integrable on $[a,b]$. This completes the proof.

Theorem 1.7.6 Let $f:[a,b]\to\mathbb{R}$ be integrable on $[a,b]$. If $\exists$ a positive real number k such that $f(x)\geq k$ for all $x\in[a,b]$ then $\frac{1}{f}$ is integrable on $[a,b]$.

Proof: $\left|\frac{1}{f}(x)\right| = \left|\frac{1}{f(x)}\right| \leq \frac{1}{k}$ for all $x\in[a,b]$. This shows that $\frac{1}{f}$ is bounded on $[a,b]$. Let us choose $\epsilon>0$. Since f is integrable on $[a,b]$, $\exists$ a partition P of $[a,b]$ such that $U(P,f)-L(P,f)<k^2\epsilon$.

Let $P=(x_0,x_1,\dots,x_n)$, where $x_0=a<x_1<\dots<x_n=b$.

Let $M_r = \sup_{x\in[x_{r-1},x_r]} f(x)$, $m_r = \inf_{x\in[x_{r-1},x_r]} f(x)$;

$M_r' = \sup_{x\in[x_{r-1},x_r]} \frac{1}{f}(x)$, $m_r' = \inf_{x\in[x_{r-1},x_r]} \frac{1}{f}(x)$, for $r=1,2,\dots,n$

For any two points $\alpha,\beta\in[x_{r-1},x_r]$, we have

$$\left|\frac{1}{f}(\alpha)-\frac{1}{f}(\beta)\right| = \left|\frac{1}{f(\alpha)}-\frac{1}{f(\beta)}\right| = \frac{1}{|f(\alpha)||f(\beta)|}\,|f(\alpha)-f(\beta)|$$

$$\leq \frac{1}{k^2}\,|f(\alpha)-f(\beta)| \quad \dots \text{(i)}$$

Since f is bounded on $[x_{r-1},x_r]$ with $\sup_{x\in[x_{r-1},x_r]} f(x) = M_r$ and $\inf_{x\in[x_{r-1},x_r]} f(x) = m_r$, the supremum of the set $\{|f(\alpha)-f(\beta)| : \alpha,\beta\in[x_{r-1},x_r]\}$ is $M_r - m_r$.

Since $\frac{1}{f}$ is bounded on $[x_{r-1},x_r]$ with $\sup_{x\in[x_{r-1},x_r]} \frac{1}{f}(x) = M_r'$

and $\inf\limits_{x\in[x_{r-1},x_r]} \frac{1}{f}(x) = m_r'$, the supremum of the set
$\left\{ \left| \frac{1}{f}(\alpha) - \frac{1}{f}(\beta) \right| : \alpha,\beta \in [x_{r-1}, x_r] \right\} = M_r' - m_r'$

It follows from inequality (i) that $\frac{1}{k^2}(M_r - m_r)$ is an upper bound of the set $\left\{ \left| \frac{1}{f}(\alpha) - \frac{1}{f}(\beta) \right| : \alpha,\beta \in [x_{r-1}, x_r] \right\}$

So, $M_r' - m_r' \leq \frac{1}{k^2}(M_r - m_r)$. This holds for $r = 1, 2, \ldots, n$

$$U\left(P, \tfrac{1}{f}\right) - L\left(P, \tfrac{1}{f}\right) = \sum_{r=1}^{n} (M_r' - m_r')(x_r - x_{r-1})$$

$$\leq \sum_{r=1}^{n} \frac{1}{k^2}(M_r - m_r)(x_r - x_{r-1})$$

$$= \frac{1}{k^2}\left[U(P,f) - L(P,f)\right] < \varepsilon$$

Therefore, for a chosen $\varepsilon > 0$, $\exists$ a partition P of $[a,b]$ such that $U\left(P, \frac{1}{f}\right) - L\left(P, \frac{1}{f}\right) < \varepsilon$

This being a sufficient condition for integrability, $\frac{1}{f}$ is integrable on $[a,b]$. This completes the proof.

<u>Note 1</u> If $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ and $f(x) > 0$ for all $x \in [a,b]$, then $\frac{1}{f}$ may not be integrable on $[a,b]$. For example, let

$$f : [0,1] \to \mathbb{R} \text{ be defined by } f(x) = \begin{cases} x, & 0 < x \leq 1 \\ 1, & x = 0 \end{cases}$$

Then f is bounded on $[0,1]$. f is continuous on $[0,1]$ except at only one point 0. So f is integrable on $[0,1]$. Also $f(x) > 0$ for all $x \in [a,b]$. $\frac{1}{f}$ is unbounded on $[0,1]$ and therefore $\frac{1}{f}$ is not integrable on $[0,1]$.