**Theorem 1.7.7** Let $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ be both integrable on $[a,b]$ and $\exists$ a positive real number $k$ such that $g(x)\geq k$ for all $x\in[a,b]$.

Then $f/g$ is integrable on $[a,b]$

Proof: Since $f\in\mathcal{R}[a,b]$, $f$ is bounded on $[a,b]$. So, $\exists$ a positive real number $B$ such that $|f(x)|<B$ for all $x\in[a,b]$. Since $g(x)\geq k$, $\forall x\in[a,b]$, $\frac{1}{g(x)}\leq\frac{1}{k}$, $\forall x\in[a,b]$. Therefore $\left|\frac{f}{g}(x)\right| = \left|\frac{f(x)}{g(x)}\right| < \frac{B}{k}$, $\forall x\in[a,b]$.

This shows that $f/g$ is bounded on $[a,b]$.

~~Sin g~~ Since $g\in\mathcal{R}[a,b]$ and $g(x)\geq k>0$, $\forall x\in[a,b]$, $\frac{1}{g}\in\mathcal{R}[a,b]$. Since $f$ and $\frac{1}{g}$ are both integrable on $[a,b]$, $\frac{f}{g}$ is integrable on $[a,b]$. by Theorem 1.7.5.

1.8

Let a function $f:[a,b]\to\mathbb{R}$ be integrable on $[a,b]$.

Then for each $x\in[a,b]$, $f$ is integrable on $[a,x]$.

So, $\int_a^x f(x)\,dx$ exists and it depends on $x$. So, we can define a function ~~$F(x)=\int_a^x f(t)\,dt$, $x\in[a,b]$~~.

$F$ on $[a,b]$ by $F(x)=\int_a^x f(t)\,dt$

**Theorem 1.8.1** If $f:[a,b]\to\mathbb{R}$ be integrable on $[a,b]$ Then the function $F$ defined by $F(x)=\int_a^x f(t)\,dt$, $x\in[a,b]$ is uniformly continuous on $[a,b]$.

Proof: Let $x_1, x_2$ be any two points in $[a,b]$.

$$F(x_2) - F(x_1) = \int_a^{x_2} f(t)\,dt - \int_a^{x_1} f(t)\,dt = \int_{x_1}^{x_2} f(t)\,dt$$

So, $$|F(x_2) - F(x_1)| = \left| \int_{x_1}^{x_2} f(t)\,dt \right|$$

Since $f$ is integrable on $[a,b]$, $f$ is bounded on $[a,b]$.
So, $\exists$ a real number $k>0$ such that $|f(x)| < k$, $\forall x \in [a,b]$. If $x_2 > x_1$, $\left| \int_{x_1}^{x_2} f(t)\,dt \right| \leq \int_{x_1}^{x_2} |f(t)|\,dt \leq (x_2 - x_1)k$

If $x_1 > x_2$ $\left| \int_{x_1}^{x_2} f(t)\,dt \right| = \left| \int_{x_2}^{x_1} f(t)\,dt \right| \leq \int_{x_2}^{x_1} |f(t)|\,dt \leq (x_1 - x_2)k$

Consequently, $|F(x_2) - F(x_1)| \leq |x_2 - x_1|\,k$

Let us choose $\varepsilon > 0$. Then $|F(x_2) - F(x_1)| < \varepsilon$
for all $x_1, x_2$ in $[a,b]$ satisfying $|x_2 - x_1| < \frac{\varepsilon}{k}$

Let $\delta = \varepsilon/k$. Then $|F(x_2) - F(x_1)| < \varepsilon$, $\forall x_1, x_2 \in [a,b]$ satisfying $|x_2 - x_1| < \delta$

This proves that $F$ is uniformly continuous on $[a,b]$

Note: As $F$ is uniformly continuous on $[a,b]$, so $F$ is continuous on $[a,b]$.

Worked Example

1. Let $f(x) = 0$, $-1 \leq x \leq 0$
$\qquad = 1$, $0 < x \leq 1$

Prove that $f$ is integrable on $[-1,1]$. Show that the function F defined by $F(x) = \int_{-1}^{x} f(t)\,dt$ is continuous on $[-1, 1]$

Solution: $f$ is bounded on $[-1, 1]$ and is continuous on $[-1, 1]$ except at only one point, 0.

So, $f$ is integrable on $[-1, 1]$

For, $-1 \le x \le 0$, $F(x) = \int_{-1}^{x} f(t)\,dt = 0$

For $0 < x \le 1$, $F(x) = \int_{-1}^{x} f(t)\,dt = \int_{-1}^{0} f(t)\,dt + \int_{0}^{x} f(t)\,dt$

$$= 0 + \int_0^x 1\,dt = x$$

So, we have $F(x) = 0, \; -1 \le x \le 0$

$$= x, \; 0 < x \le 1$$

Clearly, $F$ is continuous on $[-1, 1]$

Note: Here $f$ is not continuous on $[-1, 1]$ but $F$ is continuous on $[-1, 1]$

We observe that the function $F$ is continuous on $[a, b]$ when $f$ is integrable on $[a, b]$.

If, however $f$ is continuous on $[a, b)$ then $F$ will be differentiable on $[a, b]$ as we shall see in the next theorem.

Theorem 1.8.2 If a function $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ then the function $F$ defined by $F(x) = \int_a^x f(t)\,dt$, $x \in [a, b]$ is differentiable at any point $c \in [a, b]$ at

$f$ is continuous and $F'(c) = f(c)$

Proof: Let $c \in [a,b]$. Let $\varepsilon > 0$. Since $f$ is continuous at $c$, $\exists\ \delta > 0$ such that $|f(x) - f(c)| < \varepsilon \quad \forall x \in [c, c+\delta)$

Let us choose $h$ satisfying $0 < h < \delta$.

Then $f(c) - \varepsilon < f(x) < f(c) + \varepsilon \quad \forall x \in [c, c+h]$.

So, $\int_c^{c+h} (f(c) - \varepsilon)\,dx \leq \int_c^{c+h} f(x)\,dx \leq \int_c^{c+h} (f(c) + \varepsilon)\,dx$

or, $(f(c) - \varepsilon)h \leq F(c+h) - F(c) \leq (f(c) + \varepsilon)h$

or, $\left| \frac{F(c+h) - F(c)}{h} - f(c) \right| \leq \varepsilon$. This holds for all $h$ satisfying $0 < h < \delta$.

This implies $\lim_{h \to 0+} \frac{F(c+h) - F(c)}{h} = f(c)$

That is $RF'(c) = f(c)$ --- (i) ($RF'(c)$ is the right hand derivative)

Let $c \in [a,b]$. Let $\varepsilon > 0$. Since $f$ is continuous at $c$, $\exists$ a positive $\delta_1$ such that $|f(x) - f(c)| < \varepsilon \quad \forall x \in (c - \delta_1, c]$.

Let us choose $h$ satisfying $0 < h < \delta_1$. Then

$f(c) - \varepsilon < f(x) < f(c) + \varepsilon \quad \forall x \in [c-h, c]$

So, $\int_{c-h}^{c} (f(c) - \varepsilon)\,dx \leq \int_{c-h}^{c} f(x)\,dx \leq \int_{c-h}^{c} (f(c) + \varepsilon)\,dx$

or, $(f(c) - \varepsilon)h \leq F(c) - F(c-h) \leq (f(c) + \varepsilon)h$

or, $\left| \frac{F(c-h) - F(c)}{-h} - f(c) \right| \leq \varepsilon$. This holds for all $h$ satisfying $0 < h < \delta_1$