This implies $\lim\limits_{h \to 0-} \dfrac{F(c+h) - F(c)}{h} = f(c)$

That is $LF'(c) = f(c) \quad \cdots$ (ii) ($LF'(c)$ is the left hand derivative)

From (i) & (ii) it follows that F is differentiable at any point $c \in [a,b]$ at which f is continuous and $F'(c) = f(c)$

Corollary: If $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ then F is differentiable on $[a,b]$ and $F'(x) = f(x)$ for all $x \in [a,b]$.

Note: Here in the previous theorem, we have used many results that we have not discussed earlier. Here we mention the results only.

1. If $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ and $a < c < b$ then $f : [a,c] \to \mathbb{R}$ is integrable on $[a,c]$ and $f : [c,b] \to \mathbb{R}$ is integrable on $[c,b]$. Also $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

2. If $f : [a,b] \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$ be both integrable on $[a,b]$ and $f(x) < g(x)$ $\forall x \in [a,b]$, then
$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

3. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

4. If $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ ~~is also integrable on~~ then $\int_a^b |f(x)|\,dx \geq \left|\int_a^b f(x)\,dx\right|$

1.8.3 Definition: A function ϕ is called an antiderivative or a primitive or an indefinite integral of a function f on an interval I, if $\phi'(x) = f(x)$ $\forall x \in I$

If ϕ be an antiderivative of f on I, then $\phi + c$ where $c \in \mathbb{R}$ is obviously an antiderivative of f on I. This shows that if f admits of an antiderivative on I, then there exists many antiderivatives ~~of~~ of f on I.

It follows from the previous theorem that if f be continuous on ~~I~~ a closed interval $[a, b]$, then f possesses an antiderivative on $[a, b]$ given by F. Therefore continuity of f ensures the existence of an antiderivative of f.

Note that continuity of f is not a necessary condition for the existence of an antiderivative of f.

For example, let $f : [-1, 1] \to \mathbb{R}$ is defined by

$$f(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x}, \quad x \neq 0$$
$$= 0, \quad x = 0$$

f is not continuous on $[-1, 1]$, 0 being the point of discontinuity.

Let $\phi : [-1, 1] \to \mathbb{R}$ be defined by

$$\phi(x) = x^2 \sin\frac{1}{x}, \quad x \neq 0$$
$$= 0, \quad x = 0$$

Then $\phi'(x) = f(x)$ $\forall x \in [-1, 1]$

Thus F is an antiderivative of f on $[-1,1]$ although f is not continuous

Worked Examples (continues)

2. Let $f(x) = 1 \quad 0 \le x \le 1$
$\quad\quad\quad\; = x \quad 1 < x \le 2$

Verify that the function F defined by $F(x) = \int_0^x f(t)dt$, $x \in [0,2]$, is differentiable on $[0,2]$ and $F'(x) = f(x)$ on $[0,2]$

Solution: f is continuous on $[0,2]$ and so, integrable on $[0,2]$

Hence $\int_0^x f(t)dt$ exists for all $x \in [0,2]$

For $0 \le x \le 1$, $F(x) = \int_0^x f(t)dt = \int_0^x dt = x$

For $1 < x \le 2$, $F(x) = \int_0^x f(t)\,dt = \int_0^1 f(t)dt + \int_1^x f(t)\,dt$

$$= 1 + \int_1^x t\,dt = 1 + \left(\frac{x^2}{2} - \frac{1}{2}\right) = \frac{1}{2}(x^2+1)$$

So, we have $F(x) = x, \quad 0 \le x \le 1$

$$= \frac{1}{2}(x^2+1) \quad 1 < x \le 2$$

Now $\lim\limits_{h \to 0-} \dfrac{F(1+h) - F(1)}{h} = \lim\limits_{h \to 0-} \dfrac{1+h-1}{h} = \lim\limits_{h \to 0-} \dfrac{h}{h} = 1$

and $\lim\limits_{h \to 0+} \dfrac{F(1+h) - F(1)}{h} = \lim\limits_{h \to 0+} \dfrac{\frac{1}{2}((1+h)^2+1) - 1}{h}$

$$= \lim_{h \to 0+} \frac{1 + h + \frac{1}{2}h^2 - 1}{h} = \lim_{h \to 0+} \frac{h(1+\frac{1}{2}h)}{h}$$

$$= 1$$

Therefore, $F'(1) = 1$

So, we have $F'(x) = 1, \quad 0 \le x \le 1$

$= x, \quad 1 < x \le 2$

That is, $F'(x) = f(x)$ for all $x \in [0, 2]$

<u>Theorem 1.8.4</u> If $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and $\phi : [a, b] \to \mathbb{R}$ be an antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a)$$

Proof: Since f is continuous on $[a, b]$, f is integrable on $[a, b]$. ~~and~~ Let $F(x) = \int_a^x f(t)\,dt$, $x \in [a, b]$

Since f is continuous on $[a, b]$, F is differentiable on $[a, b]$ and $F'(x) = f(x)$ $\forall x \in [a, b]$. So, F is an antiderivative of f on $[a, b]$. Since ϕ is an antiderivative of f on $[a, b]$, for all $x \in [a, b)$, $\phi(x) = F(x) + c$, where c is a constant.

So, $\phi(a) = F(a) + c = c$, since $F(a) = 0$

So, $\phi(x) = F(x) + \phi(a)$ $\forall x \in [a, b]$

Consequently $\int_a^b f(x)\,dx = F(b) = \phi(b) - \phi(a)$.

Note: The theorem states that if f be continuous on $[a, b]$ then the integral $\int_a^b f(x)\,dx$ can be evaluated in terms of an antiderivative of f on $[a, b]$

<u>Theorem 1.8.5</u> Fundamental theorem of Integral Calculus

If (i). $f : [a, b] \to \mathbb{R}$ be integrable and
(ii) f possesses an antiderivative ϕ on $[a, b]$,

then $\int_a^b f(x)\,dx = \phi(b) - \phi(a)$