Proof: Let $P = (x_0, x_1, \dots, x_n)$ where $a = x_0 < x_1 \dots < x_n = b$ be a partition of $[a,b]$.

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$, for $r = 1, 2, \dots, n$.

Since $\phi'(x) = f(x)\ \forall x \in [a,b]$, ϕ satisfies all condition of Lagrange's mean value theorem on $[x_{r-1}, x_r]$, for $r = 1, 2, \dots, n$

Therefore for $r = 1, 2, \dots n$,

$$\phi(x_r) - \phi(x_{r-1}) = \phi'(\xi_r)(x_r - x_{r-1}) \text{ for some } \xi_r \text{ in } (x_{r-1}, x_r)$$
$$= f(\xi_r)(x_r - x_{r-1}).$$

The summation gives

$$\sum_{r=1}^{n} f(\xi_r)(x_r - x_{r-1}) = \phi(b) - \phi(a)$$

but $m_r \leq f(\xi_r) \leq M_r$ for $r = 1, 2, \dots, n$

So, $$\sum_{r=1}^{n} m_r (x_r - x_{r-1}) \leq \phi(b) - \phi(a) \leq \sum_{r=1}^{n} M_r (x_r - x_{r-1})$$

So, $L(P, f) \leq \phi(b) - \phi(a) \leq U(P, f)$

This holds for all partition P of $[a, b]$

So, $\phi(b) - \phi(a)$ is an upper bound of the set $\{L(P,f) : P \in \mathcal{P}[a,b]\}$. As the supremum of the set is $\underline{\int_a^b} f(x)dx$, it follows that

$$\underline{\int_a^b} f(x)dx \leq \phi(b) - \phi(a) \quad \dots \text{(i)}$$

As $\phi(b)-\phi(a)$ is a lower bound of the set $\{U(P,f) : P \in \mathcal{P}[a,b]\}$. As the ~~supremum~~ infimum of the set is $\overline{\int_a^b} f(x)\,dx$, it follows that

$$\overline{\int_a^b} f(x)\,dx \geq \phi(b)-\phi(a) \quad \cdots (ii)$$

From (i) and (ii) $\underline{\int_a^b} f(x)\,dx \leq \phi(b)-\phi(a) \leq \overline{\int_a^b} f(x)\,dx$

Since f is integrable on $[a,b]$, $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$
$$= \int_a^b f(x)\,dx$$

Consequently, $\int_a^b f(x)\,dx = \phi(b)-\phi(a)$

Corollary: Since $\phi(b)-\phi(a) = (b-a)f(\xi)$ for some $\xi \in (a,b)$, $\int_a^b f(x)\,dx = (b-a)\,f(a+(b-a)\theta)$ for some θ satisfying $0<\theta<1$

Note: The evaluation of the integral $\int_a^b f(x)\,dx$ in terms of the antiderivative of f is possible if f satisfies the condition of theorem. That these conditions are independent of each other can be ~~seen~~ seen from the following two examples:

(i) Let $f : [-1,1] \to \mathbb{R}$ be defined by $f(x) = 0, \; -1 \leq x < 0$
$$= 1, \; 0 \leq x \leq 1$$

f is bounded on $[-1,1]$ and continuous on $[-1,1]$ except at only one point, 0. So, f is integrable on $[-1,1]$.

If possible, let g be an antiderivative of f on $[-1, 1]$. Then $g'(x) = 0$, $-1 \le x < 0$
$= 1$, $0 \le x \le 1$

g is differentiable on $[-1, 1]$ and $g'(-1) \neq g'(1)$. By Darboux's theorem g' must every value between 0 and 1. But it does not do so. It follows that g does not exist, That is, f has no antiderivative on $[-1,1]$. Therefore f has no antiderivative on $[-1, 1]$ although f is integrable on $[-1,1]$

(ii) Let $f: [-1,1] \to \mathbb{R}$ be defined by

$$f(x) = 2x \sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2}, \quad x \neq 0$$
$$= 0, \quad x = 0$$

f is unbounded in every neighbourhood of 0.

Hence f is not integrable on $[-1, 1]$.

Let $\phi : [-1, 1] \to \mathbb{R}$ be defined by

$$\phi(x) = x^2 \sin\frac{1}{x^2}, \quad x \neq 0$$
$$= 0, \quad x = 0$$

Then $\phi'(x) = f(x)$ on $[-1, 1]$. So, ϕ is an antiderivative of f on $[-1, 1]$

So, f has antiderivative on $[-1,1]$ although f is not integrable on $[-1,1]$.

Worked Examples

3. Let f be defined on $[-2,2]$ by

$$f(x) = 3x^2 \cos\frac{\pi}{x^2} + 2\pi \sin\frac{\pi}{x^2}, \quad x \neq 0$$
$$= 0, \quad x = 0$$

Show that f is integrable on $[-2,2]$.

Evaluate $\int_{-2}^{2} f(x)\,dx$.

Solution: f is bounded on $[-2,2]$. f is continuous on $[-2,2]$ except at 0. Since f is continuous on $[-2,2]$ except at only one point, f is integrable on $[-2,2]$

Let $\phi : [-2,2] \to \mathbb{R}$ be defined by

$$\phi(x) = x^3 \cos\frac{\pi}{x^2}, \quad x \neq 0$$
$$= 0, \quad x = 0$$

Then $\phi'(x) = 3x^2 \cos\frac{\pi}{x^2} + 2\pi \sin\frac{\pi}{x^2}$, for all $x (\neq 0) \in [-2,2]$

$$\lim_{x\to 0} \frac{\phi(x) - \phi(0)}{x-0} = \lim_{x \to 0} x^2 \cos\frac{\pi}{x^2} = 0$$

So, $\phi'(0) = 0$

Hence ϕ is an antiderivative of f on $[-2,2]$

by the fundamental theorem

$$\int_{-2}^{2} f(x)\,dx = \phi(2) - \phi(-2)$$
$$= 8 \cos\frac{\pi}{4} + 8 \cos\frac{\pi}{4}$$
$$= 8\sqrt{2}$$