4. Evaluate $\lim_{x\to 0} \dfrac{\int_0^{x^2} e^{\sqrt{1+t}}\,dt}{x^2}$

Solution: Let $y = \int_0^{x^2} e^{\sqrt{1+t}}\,dt$; $u = x^2$;

$F(x) = \int_0^x e^{\sqrt{1+t}}\,dt$; $f(t) = e^{\sqrt{1+t}}$, $t \in \mathbb{R}$. Then $y = F(u)$

Since f is continuous on $[0, x]$, $F'(x) = f(x) = e^{\sqrt{1+x}}$.

now, $\dfrac{dy}{dx} = F'(u)\dfrac{du}{dx} = 2x F'(u)$

So, $\dfrac{dy}{dx} = 2x e^{\sqrt{1+x^2}}$

So, $\lim_{x\to 0} \dfrac{\int_0^{x^2} e^{\sqrt{1+t}}\,dt}{x^2}$ (form $\frac{0}{0}$)

$= \lim_{x\to 0} \dfrac{2x e^{\sqrt{1+x^2}}}{2x} = \lim_{x\to 0} e^{\sqrt{1+x^2}} = e$

1.9 Logarithmic function

Definition: The logarithmic function L (or log) is defined by $L(x) = \int_1^x \frac{1}{t}\,dt$, for $x > 0$

Property 1 $L(1) = 0$

From definition it follows that

$L(1) = \int_1^1 \frac{1}{t}\,dt = 0$

Property 2 $L(x) < 0$ if $0 < x < 1$

$= 0$ if $x = 1$

> 0 if $x > 1$

~~SS~~ Proof: If $0<x<1$, the function f defined by $f(t)=\frac{1}{t}$, $t\in[x,1]$ is continuous on $[x,1]$ and $f(t)>0$ for all $t\in[x,1]$

So, $\int_x^1 f(t)\,dt>0$. That is, $L(x)<0$

If $x>1$, the function f defined by $f(t)=\frac{1}{t}$, $t\in[1,x]$ is continuous on $[1,x]$ and $f(t)>0$ $\forall t\in[1,x]$

So, $\int_1^x f(t)\,dt>0$. That is, $L(x)>0$

So,
$$L(x) \begin{cases} <0 & \text{if } 0<x<1 \\ =0 & \text{if } x=1 \\ >0 & \text{if } x>1 \end{cases}$$

<u>Property 3</u>. For $x>0$, $y>0$ $\quad L(xy)=L(x)+L(y)$

Proof: Since $xy>0$, $L(xy)=\int_1^{xy}\frac{1}{t}\,dt$

$$=\int_1^x \frac{1}{t}\,dt+\int_x^{xy}\frac{1}{t}\,dt$$

$$=L(x)+\int_x^{xy}\frac{1}{t}\,dt=L(x)+\int_1^y\frac{1}{u}\,du$$

[putting $t=xu$

$$=L(x)+L(y)$$

<u>Corollary 1</u>. In particular if $y=\frac{1}{x}$, then

$$L(x)+L\left(\frac{1}{x}\right)=L(1)=0.$$

So, $L\left(\frac{1}{x}\right)=-L(x)$, $\quad x>0$

Corollary 2 For $x>0$, $y>0$ $L\left(\frac{x}{y}\right) = L\left(x \cdot \frac{1}{y}\right)$

$$= L(x) + L\left(\frac{1}{y}\right) = L(x) - L(y)$$

Property 4 For $x>0$, $L(x^n) = n\,L(x)$, n being an integer

Proof : Case 1: $n=0$

In this case $L(x^n) = L(1) = 0$ and $n\,L(x) = 0$

So, $L(x^n) = n\,L(x)$

Case 2 n is a positive integer

When $n=1$, the property holds

Let the property holds for $n=m$, where m is a positive integer.

Then $L(x^m) = m\,L(x)$

So, $L(x^{m+1}) = L(x^m) + L(x)$, by property 3

$$= m\,L(x) + L(x)$$

$$= (m+1)\,L(x)$$

This shows that the property holds for $n = m+1$ if it holds for $n=m$. Also the property holds for $n=1$. By principle of mathematical induction, the property holds for all positive integer n

Case 3 n is a negative integer

Let $n = -m$, where m is a positive integer

Then $L(x^n) = L(x^{-m}) = L\left(\left(\frac{1}{x}\right)^m\right)$

$= m\,L\left(\frac{1}{x}\right)$ by Case 2

$= -m\,L(x)$

$= n\,L(x)$

Combining all cases, the proof is complete.

Property 5. For $x > 0$, $L(x^\alpha) = \alpha L(x)$, α being a rational number.

Proof: Case 1, α is an integer

This is property 3

Case 2. α is a positive fraction

Let $\alpha = p/q$, p and q are positive integers, $q > 1$.

$L(x^\alpha) = L(x^{p/q}) = L\left(\left(x^{1/q}\right)^p\right)$

$= p\,L(x^{1/q})$, by property 3

Also $L(x) = L\left(\left(x^{1/q}\right)^q\right) = q\,L(x^{1/q})$ by property 3

Therefore, $L(x^\alpha) = \frac{p}{q} L(x) = \alpha L(x)$

Case 3 α is a negative fraction

Let $\alpha = -\beta$, where β is a positive fraction