Then $L(x^{\alpha}) = L(x^{-\beta}) = L\left(\left(\frac{1}{x}\right)^{\beta}\right) = \beta L\left(\frac{1}{x}\right)$ by Case 2

$= -\beta L(x) = \alpha L(x)$

Combining all these case, the proof is complete.

Property 6 The function L defined by $L(x) := \int_1^x \frac{1}{t}\,dt$, $x > 0$ is continuous on $(0, \infty)$

Proof: As $L(x)$ is defined as an integral, it follows from Theorem 1.8.1 that L is continuous on $(0, \infty)$

Corollary: If $x > 0$ and α be a real number,

$L(x^{\alpha}) = \alpha L(x)$

Proof: Let α be irrational, let us consider a sequence $\{\alpha_n\}$ of rational points converging to α.

Then $L(x^{\alpha_n}) = \alpha_n L(x) \quad \forall n \in \mathbb{N}$

Taking limit as $n \to \infty$ and noting that L is continuous, we have $L(x^{\alpha}) = \alpha L(x)$

Property 7 $\frac{d}{dx} L(x) = \frac{1}{x}$, $x > 0$

Proof: Let $x_0 > 0$ and let us choose $h > 0$

Then $\frac{L(x_0+h) - L(x_0)}{h} = \frac{1}{h} \int_{x_0}^{x_0+h} \frac{1}{t}\,dt$

For all $t \in [x_0, x_0+h]$, $\frac{1}{x_0+h} \le \frac{1}{t} \le \frac{1}{x_0}$

So, we have $\int_{x_0}^{x_0+h} \frac{1}{x_0+h}\,dt \le \int_{x_0}^{x_0+h} \frac{1}{t}\,dt \le \int_{x_0}^{x_0+h} \frac{1}{x_0}\,dt$

or, $\frac{1}{x_0+h} \leq \frac{L(x_0+h)-L(x_0)}{h} \leq \frac{1}{x_0}$

By Sandwich theorem, $\lim_{h\to 0+} \frac{L(x_0+h)-L(x_0)}{h} = \frac{1}{x_0}$ ---(i)

Let us choose $h<0$ such that $x_0+h>0$

For all $t \in [x_0+h, x_0]$, $\frac{1}{x_0} \leq \frac{1}{t} \leq \frac{1}{x_0+h}$

So, we have $\int_{x_0+h}^{x_0} \frac{1}{x_0}\,dt \leq \int_{x_0+h}^{x_0} \frac{1}{t}\,dt \leq \int_{x_0+h}^{x_0} \frac{1}{x_0+h}\,dt$

or, $\frac{-h}{x_0} \leq \int_{x_0+h}^{x_0} \frac{1}{t}\,dt \leq \frac{-h}{x_0+h}$

or, $\frac{1}{x_0} \leq \frac{1}{h}\int_{x_0}^{x_0+h} \frac{1}{t}\,dt \leq \frac{1}{x_0+h}$.

By Sandwich theorem, $\lim_{h\to 0-} \frac{L(x_0+h)-L(x_0)}{h} = \frac{1}{x_0}$ ---(ii)

From (i) and (ii), we have $\lim_{h\to 0} \frac{L(x_0+h)-L(x_0)}{h} = \frac{1}{x_0}$

This implies $\frac{d}{dx}L(x) = \frac{1}{x}$, $x>0$

Corollary $\lim_{x\to 0} \frac{L(1+x)}{x} = 1$

Property 8 The logarithmic function L defined by $L(x) = \int_1^x \frac{1}{t}\,dt$, $x>0$ is a bijective function from $(0,\infty)$ to $(-\infty,\infty)$

Proof: As L is strictly increasing function on $(0,\infty)$, it is one-to-one on $(0,\infty)$

As $L(x)\to\infty$ as $x\to\infty$ and $L(x)\to-\infty$ as $x\to 0+$ and L is continuous and strictly increasing in $(0,\infty)$, the function L assumes every real number in $(-\infty,\infty)$ exactly once. This proves that L is a bijective function with domain $(0,\infty)$ and range $(-\infty,\infty)$

Definition. The unique real number x satisfying $L(x)=1$ is denoted by e, i.e $L(e)=1$

Therefore e is defined by

$$1=\int_1^e \frac{1}{t}\,dt$$

<u>Property 9</u> $e=\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

Proof: We have $\frac{d}{dx}L(x)=\frac{1}{x}$, $x>0$. So,

$L'(1)=1$

That is $\lim\limits_{h\to 0}\dfrac{L(1+h)-L(1)}{h}=1$

or, $\lim\limits_{h\to 0}\dfrac{L(1+h)}{h}=1$

Let us consider a sequence $\{h_n\}$ where $h_n=\frac{1}{n}$, $\lim h_n=0$. By sequential criterion $\lim\limits_{h_n\to 0}\dfrac{L(1+h_n)}{h_n}=1$

or, $\lim\limits_{n\to\infty} n\,L\left(1+\frac{1}{n}\right)=1$

or, $\lim\limits_{n\to\infty} L\left\{\left(1+\frac{1}{n}\right)^n\right\}=1$

or, $L\left\{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right\}=1$, since L is continuous

Since $L(e) = 1$ and L is a bijective function on $(0, \infty)$, it follows that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

1.10 Mean Value Theorem

1.10.1 Theorem (First Mean Value Theorem)

If (i) $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ be both integrable on $[a,b]$ and

(ii) $g(x)$ has the same sign for all $x\in[a,b]$

then there is a number μ such that

$$\int_a^b f(x)g(x)\,dx = \mu\int_a^b g(x)\,dx \quad \text{where } m \le \mu \le M$$

and $m = \inf\limits_{x\in[a,b]} f(x)$ and $M = \sup\limits_{x\in[a,b]} f(x)$

~~Proof: Case I Let $g(x) > 0$, $x \in [a,b]$~~

If further, f is continuous on $[a,b]$, then $\exists$ a point ξ in $[a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx$$

Proof: Case I Let $g(x) > 0$, $x\in[a,b]$

Since $m = \inf\limits_{x\in[a,b]} f(x)$ and $M = \sup\limits_{x\in[a,b]} f(x)$, $m \le f(x) \le M$ $\forall x\in[a,b]$

So, $m\,g(x) \le f(x)g(x) \le M\,g(x)$ $\forall x\in[a,b]$

Since f and g are both integrable on $[a,b]$, mg, fg and Mg are all integrable on $[a,b]$

and $\int_a^b m\,g(x)\,dx \le \int_a^b f(x)g(x)\,dx \le \int_a^b M g(x)\,dx$