or, $m \int_a^b g(x)dx \le \int_a^b f(x)g(x) \le M \int_a^b g(x)dx$

So, $\int_a^b f(x)g(x)dx = \mu \int_a^b g(x)dx$, where $m \le \mu \le M$

Case 2 $g(x) < 0$, $x \in [a,b]$

The proof is similar.

<u>Second part</u> If f be continuous on $[a,b]$ $\exists$ a point ξ in $[a,b]$ such that $f(\xi) = \mu$, where $m \le \mu \le M$

It follows that $\int_a^b f(x)g(x)dx = f(\xi) \int_a^b g(x)dx$, where $a \le \xi \le b$

This completes the proof.

<u>Examples</u>

1. If $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and $\int_a^b f(x)dx = 0$, prove that $\exists$ at least one point $c \in [a,b]$ such that $f(c) = 0$

Solution: Since f is continuous on $[a,b]$, f is integrable on $[a,b]$. By the first Mean Value theorem $\exists$ a point c in $[a,b]$ such that $\int_a^b f(x)dx = f(c)(b-a)$.
Since $\int_a^b f(x)dx = 0$, it follows that $f(c) = 0$

2. Use first mean value theorem to prove that

$$\frac{\pi}{6} \le \int_0^{1/2} \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}} \le \frac{\pi}{6} \frac{1}{\sqrt{1-k^2/4}}, \quad k^2 < 1$$

Solution: Let $f(x) = \frac{1}{\sqrt{1-k^2x^2}}$, $x \in [0, \frac{1}{2}]$;

$g(x) = \frac{1}{\sqrt{1-x^2}}$, $x \in [0, \frac{1}{2}]$. Then f and g are integrable on $[0, \frac{1}{2}]$ and $g(x) > 0$ $\forall x \in [0, \frac{1}{2}]$

Since f is continuous on $[0, \frac{1}{2}]$, by first mean value theorem $\exists$ a point ξ in $[0, \frac{1}{2}]$ such that

$$\int_0^{1/2} f(x) g(x)\, dx = f(\xi) \int_0^{1/2} g(x)\, dx$$

or, $$\int_0^{1/2} \frac{1}{\sqrt{(1-k^2x^2)(1-x^2)}}\, dx = \frac{1}{\sqrt{1-k^2\xi^2}} \cdot \int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\, dx = \frac{\pi}{6} \frac{1}{\sqrt{1-k^2\xi^2}}$$

Since $0 \le \xi \le \frac{1}{2}$,

$$1 \le \frac{1}{\sqrt{1-k^2\xi^2}} \le \frac{1}{\sqrt{1-k^2/4}}$$

So, $$\pi/6 \le \int_0^{1/2} \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}} \le \frac{\pi}{6} \frac{1}{\sqrt{1-k^2/4}}$$

---

2. Improper Integrals

2.1 In the preceding chapter of Riemann Integral, the theory of integration was developed under two assumptions —

(i) the interval of integration was required to be a closed and bounded interval, and

(ii) the integrand was required to be bounded on the interval.

The scope of the theory of integration may be widened by relaxing these restrictions. If these restrictions are relaxed we have the following two types of integrals, called improper integrals or infinite integrals –

(a) improper integrals on a finite interval where the integrand is unbounded

(b) improper integrals on an unbounded interval.

We define convergence of improper integrals and discuss the properties of each type separately.

~~2.2. Definitions~~

~~I. Convergence of the improper integral $\int_a^b f(x)\,dx$ when a is the~~

A. Improper integrals on a closed and bounded interval, the integrand having infinite discontinuities

2.2 Definitions

I. Convergence of the improper integral $\int_a^b f(x)\,dx$ when a is the only point of infinite discontinuity of f in $[a, b]$

Let the left end point a of $[a, b]$ be the only point of infinite discontinuity of a function f which is bounded and integrable on $[a+\epsilon, b]$ for every ϵ satisfying $0 < \epsilon < b-a$

Let $\phi(\epsilon) = \int_{a+\epsilon}^b f(x)\,dx$, $0 < \epsilon < b-a$

If $\lim_{\epsilon \to 0+} \phi(\epsilon)$ exists (finitely) then the improper integral $\int_a^b f(x)\,dx$ is said to be convergent. If the limit be l, we write $\int_a^b f(x)\,dx = l$.

If $\lim_{\epsilon \to 0+} \phi(\epsilon)$ does not exist (finitely) then the improper integral is divergent.

Note. If a be the only point of infinite discontinuity of a function f which is bounded and integrable on $[a+\varepsilon, b]$ for every ε satisfying $0 < \varepsilon < b-a$ and $\int_a^b f(x)\,dx$ is convergent, then $\int_a^c f(x)\,dx$ is also convergent for all $c \in (a, b)$.

Examples 1. The integral $\int_0^1 \frac{1}{x}\,dx$ is improper, since 0 is the only point of infinite discontinuity of the integrand. The integrand is bounded and integrable on $[0+\varepsilon, 1]$ for all ε satisfying $0 < \varepsilon < 1$

$$\lim_{\varepsilon \to 0} \int_{0+\varepsilon}^1 \frac{1}{x}\,dx = \lim_{\varepsilon \to 0+} [-\log \varepsilon] = \infty$$

So, $\int_0^1 \frac{1}{x}\,dx$ is divergent.

2. The integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is improper, since 0 is the only point of infinite discontinuity of the integrand. The integrand is bounded and integrable on $[0+\varepsilon, 1]$ for all ε satisfying $0 < \varepsilon < 1$

$$\lim_{\varepsilon \to 0} \int_{0+\varepsilon}^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon \to 0+} [2 - 2\sqrt{\varepsilon}] = 2$$

So, the improper integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is convergent and $\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$

II Convergence of the improper integral $\int_a^b f(x)\,dx$ where b is the only point of infinite discontinuity of f on $[a, b]$

Let the right end point of $[a, b]$ be the only point of infinite discontinuity of a function f which is bounded and integrable on $[a, b-\varepsilon]$ for every ε satisfying $0 < \varepsilon < b-a$. Let $\phi(\varepsilon) = \int_a^{b-\varepsilon} f(x)\,dx$, $0 < \varepsilon < b-a$