If $\lim_{\varepsilon \to 0+} \phi(\varepsilon)$ exists (finitely) then the improper integral $\int_a^b f(x)\,dx$ is said to be convergent. If the limit be l, we write $\int_a^b f(x)\,dx = l$. If $\lim_{\varepsilon \to 0+} \phi(\varepsilon)$ does not exist (finitely) then the improper integral $\int_a^b f(x)\,dx$ is said to be divergent

Note: If b is the only point of infinite discontinuity of a function f which is bounded and integrable on $[a, b-\varepsilon]$ for every ε satisfying $0<\varepsilon<b-a$ and $\int_a^b f(x)\,dx$ is convergent, then $\int_c^b f(x)\,dx$ is also convergent for all $c \in (a,b)$

Example

3. The integral $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$ is improper, since 1 is a point of infinite discontinuity of the integrand. The integrand is bounded and integrable in $[0, 1-\varepsilon]$ for all ε satisfying $0<\varepsilon<1$.

$\lim_{\varepsilon \to 0+} \int_0^{1-\varepsilon} \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{\varepsilon \to 0+} \left[\sin^{-1}(1-\varepsilon)\right] = \pi/2$. So the improper integral is convergent and $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \pi/2$

III Convergence of the improper integral $\int_a^b f(x)\,dx$ when a and b are the only points of infinite discontinuity of f in $[a,b]$

Let the end points a, b of $[a,b]$ be the only points of infinite ~~discontinuit~~ discontinuity of a function f which is bounded and integrable on $[a+\varepsilon, b-\varepsilon']$ for every $\varepsilon, \varepsilon'$ satisfying $0<\varepsilon<b-a$, $0<\varepsilon'<b-a$.

Let $c \in (a,b)$. If the improper integral $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ be both convergent according to the definition given above, then the improper integral $\int_a^b f(x)\,dx$ is said to be convergent and we write

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Note: If the end points a and b of $[a,b]$ be the only points of infinite discontinuity of a function f and the improper integral $\int_a^b f(x)\,dx$ is convergent, then for any $d \in (a,b)$, $\int_a^b f(x)\,dx = \int_a^d f(x)\,dx + \int_d^b f(x)\,dx$

Example 4. The integral $\int_0^2 \frac{1}{\sqrt{x(2-x)}}\,dx$ is improper, since 0 and 2 are points of infinite discontinuity of the integrand. The integrand is bounded and integrable on $[0+\varepsilon, 2-\varepsilon']$ $\forall\, \varepsilon, \varepsilon'$ satisfying $0<\varepsilon<2$, $0<\varepsilon'<2$. Now $\lim\limits_{\varepsilon\to 0+}\int_{0+\varepsilon}^{1}\frac{1}{\sqrt{x(2-x)}}\,dx = \lim\limits_{\varepsilon\to 0+}\left[\sin^{-1}(x-1)\right]_{\varepsilon}^{1} = \pi/2$

and $\lim\limits_{\varepsilon'\to 0+}\int_{1}^{2-\varepsilon'}\frac{1}{\sqrt{x(2-x)}}\,dx = \lim\limits_{\varepsilon'\to 0}\left[\sin^{-1}(x-1)\right]_{1}^{2-\varepsilon'} = \pi/2$

So, $\int_0^2 \frac{dx}{\sqrt{x(2-x)}}$ is convergent and $\int_0^2\frac{dx}{\sqrt{x(2-x)}} = \pi/2 + \pi/2 = \pi$

IV Convergence of the improper integral $\int_a^b f(x)\,dx$ when an interior point c is the only point of infinite discontinuity of f in $[a,b]$

If the improper integrals $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ be both convergent as defined above, then $\int_a^b f(x)\,dx$ is said to be convergent and we write $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

So, if both the limits $\lim\limits_{\varepsilon\to 0+}\int_a^{c-\varepsilon} f(x)\,dx$ and $\lim\limits_{\varepsilon'\to 0}\int_{c+\varepsilon'}^{b} f(x)\,dx$ exist, then $\int_a^b f(x)\,dx = \lim\limits_{\varepsilon\to 0+}\int_a^{c-\varepsilon} f(x)\,dx + \lim\limits_{\varepsilon'\to 0}\int_{c+\varepsilon'}^{b} f(x)\,dx$

If the improper integral $\int_a^b f(x)\,dx$ is convergent, its value is also equal to the symmetric limit $\left[\lim\limits_{\varepsilon\to 0+}\int_a^{c-\varepsilon} f(x)\,dx + \lim\limits_{\varepsilon\to 0+}\int_{c+\varepsilon}^{b} f(x)\,dx\right]$

It may happen that the improper integral $\int_a^b f(x)\,dx$ is divergent but the limit $\lim\limits_{\varepsilon\to 0}\left[\int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon}^{b} f(x)\,dx\right]$ exists, then it is called the Cauchy principal value of the improper integral $\int_a^b f(x)\,dx$ and it is denoted by $P\int_a^b f(x)\,dx$.

For example, let us consider the improper integral

$\int_{-1}^{1}\frac{1}{x}\,dx$. Here $\lim\limits_{\varepsilon\to 0+}\int_{-1}^{0-\varepsilon}\frac{1}{x}\,dx + \lim\limits_{\varepsilon'\to 0+}\int_{0+\varepsilon'}^{1}\frac{1}{x}\,dx$

$= \lim\limits_{\varepsilon\to 0+}\log\varepsilon + \lim\limits_{\varepsilon'\to 0}(-\log\varepsilon')$ and this limit does not exist if $\varepsilon \to 0+$, $\varepsilon' \to 0+$ independently. But

$$\lim_{\varepsilon \to 0}\left[\int_{-1}^{0-\varepsilon}\frac{1}{x}dx + \int_{0+\varepsilon}^{1}\frac{1}{x}dx\right]$$

$$= \lim_{\varepsilon \to 0}\left[\log\varepsilon - \log\varepsilon\right] = 0$$

So, $\int_{-1}^{1}\frac{1}{x}dx$ is divergent but $P\int_{-1}^{1}\frac{1}{x}dx = 0$

V. Convergence of the improper integral $\int_a^b f(x)dx$ when a finite number of points in $[a,b)$ are the only points of infinite discontinuity of f in $[a,b]$. Let the points be $c_1, c_2, \ldots, c_m$

Case 1 Let $a < c_1 < c_2 \cdots < c_m < b$

If the improper integrals $\int_a^{c_1} f(x)dx, \int_{c_1}^{c_2} f(x)dx, \ldots, \int_{c_m}^{b} f(x)dx$ be all convergent according to the previous definition, then $\int_a^b f(x)dx$ is said to be convergent and we write

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \cdots + \int_{c_m}^{b} f(x)dx$$

Case 2 Either $a = c_1$ or $b = c_m$ or both

If $a = c_1$ then $\int_a^b f(x)dx = \int_a^{c_2} f(x)dx + \int_{c_2}^{c_3} f(x)dx + \cdots \int_{c_m}^{b} f(x)dx$ provided all the integrals on right are convergent. If $b = c_m$, then

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \cdots + \int_{c_{m-1}}^{b} f(x)dx$$, provided all the integrals on the right are convergent.

2.3 Test for convergence, positive integrand

Theorem 2.2.3.1 Let a be the only point of infinite discontinuity of a function f which is integrable on $[a+\varepsilon, b]$ for all ε satisfying $0 < \varepsilon < b-a$ and $f(x) > 0$ $\forall x \in (a,b]$. A necessary and sufficient condition for the convergence of the improper integral $\int_a^b f(x)dx$ is that $\exists$ a positive real number k such that

$$\int_{a+\varepsilon}^{b} f(x)dx < k \quad \forall\ \varepsilon \text{ satisfying } 0 < \varepsilon < b-a$$

Theorem 2.3.2 Let ~~De~~ b be the only point of infinite discontinuity of a function f which is integrable on $[a, b-\varepsilon]$ $\forall \varepsilon$ satisfying $0<\varepsilon<b-a$ and $f(x)>0$ $\forall x \in [a,b)$. A necessary and sufficient condition for the convergence of the improper integral $\int_a^b f(x)\,dx$ is that $\exists$ a positive real number k such that

$$\int_a^{b-\varepsilon} f(x)\,dx < k \quad \forall \varepsilon \text{ satisfying } 0<\varepsilon<b-a$$

Theorem 2.3.3 Comparison Test : Let a be the only point of infinite discontinuity of the functions f and g which are both integrable on $[a+\varepsilon, b]$ for all ε satisfying $0<\varepsilon<b-a$ and $0<f(x)\le k\,g(x)$ $\forall x \in (a,b]$, where $k>0$. Then

(i) $\int_a^b g(x)\,dx$ is convergent $\Rightarrow \int_a^b f(x)\,dx$ is convergent

(ii) $\int_a^b f(x)\,dx$ is divergent $\Rightarrow \int_a^b g(x)\,dx$ is divergent

Theorem 2.3.4 Comparison test (limit form) : Let a be the only point of infinite discontinuity of the functions f and g which are both integrable on $[a+\varepsilon, b]$ $\forall \varepsilon$ satisfying $0<\varepsilon<b-a$ and $f(x)>0$, $g(x)>0$, $\forall x \in (a,b]$. If $\lim_{x\to a+} \frac{f(x)}{g(x)} = l$, where l is a non-zero finite number, then the two ~~integral~~ improper integrals $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ converge or diverge together

Note: 1. If $\lim_{x\to a+0} \frac{f(x)}{g(x)} = 0$. Then $\int_a^b g(x)\,dx$ converges $\Rightarrow \int_a^b f(x)\,dx$ converges

2. If $\lim_{x\to a+} \frac{f(x)}{g(x)} = \infty$ - Then $\int_a^b g(x)\,dx$ diverges $\Rightarrow \int_a^b f(x)\,dx$ diverges.

2.3.5 A useful comparison integral :

The integral $\int_a^b \frac{dx}{(x-a)^\mu}$ is convergent if and only if $\mu<1$

2.3.6 μ-test : Let a be the only point of infinite discontinuity of a function f which is integrable on $[a+\varepsilon, b]$ for all ε in $0<\varepsilon<b-a$ and $f(x)>0$ $\forall x \in (a,b]$. If $\lim_{x\to a+} f(x)(x-a)^\mu = l$, where l is a non-zero finite number, then $\int_a^b f(x)\,dx$ is convergent if and only if $\mu<1$