Worked Examples

1. Examine the convergence of $\int_0^1 \frac{x^{p-1}}{1+x}\,dx$

Solution: The integral is a proper one if $p-1 \geq 0$. If $p<1$, 0 is the only point of discontinuity of the integrand. Let $f(x) = \frac{x^{p-1}}{1+x}$, $x \in (0,1]$, $g(x) = x^{p-1}$, $x \in (0,1]$

Then $f(x) > 0$, $g(x) > 0$ $\forall x \in (0,1]$

$\lim_{x\to 0} \frac{f(x)}{g(x)} = 1$ (a non-zero finite number) and $\int_0^1 g(x)\,dx$ is convergent if and only if $1-p<1$ i.e., if and only if $p>0$. So, $\int_0^1 \frac{x^{p-1}}{1+x}\,dx$ is convergent if and only if $p>0$

2. Show that $\int_0^1 \frac{1}{(x+1)(x+2)\sqrt{x(1-x)}}\,dx$ is convergent

Solution: Let the given integral be $\int_0^1 f(x)\,dx$. 0 and 1 are the only points of infinite discontinuity of f. $f(x) > 0$ $\forall x \in (0,1)$. Let us examine the convergence of the improper integral $\int_0^{1/2} f(x)\,dx$ and $\int_{1/2}^1 f(x)\,dx$

Convergence of $\int_0^{1/2} f(x)\,dx$ at 0

$\lim_{x\to 0+} \sqrt{x}\, f(x) = \frac{1}{2}$. By μ test, $\int_0^{1/2} f(x)\,dx$ is convergent ... (i)

Convergence of $\int_{1/2}^1 f(x)\,dx$ at 1.

$\lim_{x\to 1-} \sqrt{1-x}\, f(x) = \frac{1}{6}$. By μ test, $\int_{1/2}^1 f(x)\,dx$ is convergent. ... (ii)

From (i) and (ii) it follows that

$$\int_0^1 \frac{1}{(x+1)(x+2)\sqrt{x(1-x)}}\,dx \text{ is convergent.}$$

3. Show that $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is convergent if and only if m, n are both positive.

Proof: Let the given integral be $\int_0^1 f(x)\,dx$. It is a proper integral if $m \geq 1$ and $n \geq 1$. 0 is the only point of infinite discontinuity if $m < 1$ and 1 is the only point of infinite discontinuity if $n < 1$

Let us examine the convergence of $\int_0^{1/2} x^{m-1}(1-x)^{n-1}\,dx$ when $m < 1$ and the convergence of $\int_{1/2}^1 x^{m-1}(1-x)^{n-1}\,dx$ when $n < 1$

Convergence of $\int_0^{1/2} x^{m-1}(1-x)^{n-1}\,dx$ at 0 when $m < 1$

$f(x) > 0 \; \forall x \in (0, 1/2]$. $\lim_{x \to 0+} f(x)\, x^{1-m} = \lim_{x \to 0+} (1-x)^{n-1} = 1$ (a non-zero finite number). By μ test, $\int_0^{1/2} f(x)\,dx$ is convergent if and only if $1-m < 1$ i.e., if and only if $m > 0$

Convergence of $\int_{1/2}^1 x^{m-1}(1-x)^{n-1}\,dx$ at 1 when $n < 1$

$f(x) > 0 \; \forall x \in [1/2, 1)$. $\lim_{x \to 1-} f(x)(1-x)^{1-n} = \lim_{x \to 1-} x^{m-1} = 1$ (a non-zero finite number). By μ test, $\int_{1/2}^1 f(x)\,dx$ is convergent if and only if $1-n < 1$, i.e., if and only if $n > 0$.

Therefore both the integrals $\int_0^{1/2} x^{m-1}(1-x)^{n-1}\,dx$ and $\int_{1/2}^1 x^{m-1}(1-x)^{n-1}\,dx$ are convergent if and only if $m > 0$ and $n > 0$

Hence $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is convergent if and only if $m > 0$ and $n > 0$

Note. The integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, $m > 0$, $n > 0$ is called the Beta function and is denoted by $B(m, n)$.

Definition: The improper integral $\int_a^b f(x)\,dx$ is said to be absolutely convergent if $\int_a^b |f|(x)\,dx$ is convergent

Theorem 2.4.1 An absolutely convergent improper integral $\int_a^b f(x)\,dx$ (where a is the only point of infinite discontinuity of f in $[a,b)$ and f is integrable in $[a+\varepsilon, b]$ for all ε satisfying $0<\varepsilon<b-a$) is convergent.

Note 1 The converse of the theorem is not true. We shall establish this by some example.

Note 2 Since $|f(x)|$ is always positive, comparison test can be applied to establish the convergence of the improper integral $\int_a^b |f|(x)\,dx$

Worked Examples (contd.)

4. Show that the improper integral $\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}\,dx$ is convergent

Proof: Let $f(x) = \frac{\sin\frac{1}{x}}{\sqrt{x}}$, $x \in (0,1]$

0 is the only point of infinite discontinuity of f.

$f(x)$ does not keep the same sign in the interval $(0,1]$

$\left|\frac{\sin\frac{1}{x}}{\sqrt{x}}\right| \le \frac{1}{\sqrt{x}}$ $\forall x \in (0,1]$ and $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is convergent

So, $\int_0^1 \left|\frac{\sin\frac{1}{x}}{\sqrt{x}}\right| dx$ is convergent, i.e., $\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}\,dx$ is absolutely convergent and so, $\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}$ is convergent.

5. A function f is defined on $[0,1]$ by $f(0)=0$

$f(x) = (-1)^{n+1}(n+1)$, for $\frac{1}{n+1} < x \le \frac{1}{n}$ $(n=1,2,3,\ldots)$

Examine the convergence of the integral (i) $\int_0^1 f(x)\,dx$ (ii) $\int_0^1 |f|(x)\,dx$

Solution: (i) f is bounded and integrable on $[\epsilon, 1]$ for every $\epsilon > 0$. 0 is the only point of infinite discontinuity of f in $[0,1]$.
Let us choose $\epsilon > 0$. $\exists$ a natural number p such that

$$\frac{1}{p+1} < \epsilon \le \frac{1}{p}.$$

$$\int_\epsilon^1 f(x)\,dx = \int_{1/2}^1 f(x)\,dx + \int_{1/3}^{1/2} f(x)\,dx + \dots + \int_{1/p}^{1/p-1} f(x)\,dx + \int_\epsilon^{1/p} f(x)\,dx$$

$$= \int_{1/2}^1 2\,dx + \int_{1/3}^{1/2}(-3)\,dx + \dots + \int_{1/p}^{1/p-1} (-1)^p p\,dx + \int_\epsilon^{1/p} (-1)^{p+1}(p+1)\,dx$$

$$= 1 - \tfrac12 + \tfrac13 - \dots + (-1)^p \frac{1}{p-1} + (-1)^{p+1} \int_\epsilon^{1/p} (p+1)\,dx$$

$$\left| \int_\epsilon^1 f(x)\,dx - \left[1 - \tfrac12 + \tfrac13 - \dots + (-1)^p \frac{1}{p-1}\right] \right| = \left| (p+1) \int_\epsilon^{1/p} dx \right|$$

$$< \frac{1}{p}, \text{ since } \int_\epsilon^{1/p} dx < \int_{\frac{1}{p+1}}^{1/p} dx$$

As $\epsilon \to 0$, $p \to \infty$

So, $$\lim_{\epsilon\to 0} \int_\epsilon^1 f(x)\,dx = \lim_{p\to\infty} \left[1 - \tfrac12 + \tfrac13 - \dots + (-1)^p \frac{1}{p-1}\right] \quad \dots (i)$$

Since the series $\left[1 - \frac12 + \frac13 - \dots\right]$ is a convergent series, it follows from (i) that $\lim_{\epsilon\to 0} \int_\epsilon^1 f(x)\,dx$ is finite and therefore $\int_0^1 f(x)\,dx$ is convergent.

(ii) $|f|$ is bounded and integrable on $[\epsilon, 1]$ for every $\epsilon > 0$. 0 is the only point of infinite discontinuity of $|f|$ in $[0,1]$
Let us choose $\epsilon > 0$. $\exists$ a natural number p such that $\frac{1}{p+1} < \epsilon \le \frac{1}{p}$.

$$\int_\epsilon^1 |f|(x)\,dx = \int_{1/2}^1 |f|(x)\,dx + \int_{1/3}^{1/2} |f|(x)\,dx + \dots + \int_{1/p}^{1/p-1} |f|(x)\,dx + \int_\epsilon^{1/p} |f|(x)\,dx$$

$$= \int_{1/2}^1 2\,dx + \int_{1/3}^{1/2} 3\,dx + \dots \int_{1/p}^{1/p-1} p\,dx + \int_\epsilon^{1/p} (p+1)\,dx$$

$$= 1 + \tfrac12 + \tfrac13 + \dots + \frac{1}{p-1} + (p+1)\left(\frac1p - \epsilon\right)$$

$$> 1 + \tfrac12 + \tfrac13 + \dots + \frac{1}{p-1} \quad \text{since } \frac1p > \epsilon \quad \dots (ii)$$