Notes of CORE COURSE - ~~VII~~ 8 (SEMESTER- ~~III~~ IV) Page-1

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Riemann Integration:

1.1 Partition : Let $[a,b]$ be a closed and bounded interval. A partition of $[a,b]$ is a finite ordered set $P = \{x_0, x_1, \ldots, x_n\}$ (sometimes denoted by $(x_0, x_1, \ldots, x_n)$) of points of $[a,b]$ s.t $a = x_0 < x_1 < x_2 \cdots < x_n = b$.

The family of all partitions of $[a,b]$ is denoted by $\mathcal{P}[a,b]$

Example 1.1.1(a) $P = (0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1)$ is a partition of $[0,1]$

(b) $Q = (0, \frac{1}{4}, \frac{3}{8}, \frac{3}{4}, \frac{7}{8}, 1)$ is another partition of $[0,1]$

(c) $P_1 = (0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1)$ is a partition of $[0,1]$

$P_2 = (0, \frac{1}{2n}, \frac{2}{2n}, \ldots, \frac{2n-1}{2n}, 1)$ is a partition of $[0,1]$

$P_3 = (0, \frac{1}{3n}, \frac{2}{3n}, \ldots, \frac{3n-1}{3n}, 1)$ is a partition of $[0,1]$

(d) $R = (0, \frac{1}{4}, \frac{3}{4})$ is not a partition of $[0,1]$ as $1 \notin R$.

1.2 Refinement of a Partition : Let $[a,b]$ be a closed and bounded interval. Let P and Q be two partitions of $[a,b]$. Q is said to be a refinement of P if P be a proper subset of Q

Example 1.2.1 . $P = (0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1)$ is a partition of $[0,1]$.
and $Q = (0, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, 1)$ is a partition of $[0,1]$
Here Q is a refinement of P as P is a proper subset of Q i.e $P \subsetneq Q$. Let $R = (0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, 1)$. Then R is a refinement of P but Q is not a refinement of R

Note: If P and Q are two partitions of $[a,b]$. Then $P \cup Q$ is a common refinement of P and Q.

1.3. Upper Darboux sum $U(P,f)$ and Lower Darboux sum $U(P,f)$ and associated results:

Let $[a,b]$ be a closed and bounded interval. Let $f : [a,b] \to \mathbb{R}$ be a bounded function on $[a,b]$. Let $P = (x_0, x_1, \ldots x_n)$ be a partition of $[a,b]$ $\therefore$ $a = x_0 < x_1 < x_2 \cdots < x_n = b$.

Since f is bounded on $[a,b]$; f is bounded on $[x_{r-1}, x_r]$, for $r = 1, 2 \ldots n$.

Let $M = \sup\limits_{x \in [a,b]}$, $m = \inf\limits_{x \in [a,b]} f(x)$;

$$M_r = \sup_{x \in [x_{r-1}, x_r]} f(x), \quad m_r = \inf_{x \in [x_{r-1}, x_r]} f(x), \quad r = 1, 2, \ldots, n.$$

Then $m \le m_r \le M_r \le M$ for $r = 1, 2, \ldots, n$ ... (i)

The upper Darboux sum or the upper sum of f corresponding to the partition P, denoted by $U(P,f)$, is defined as $U(P,f) = \sum\limits_{r=1}^{n} M_r(x_r - x_{r-1})$

The lower Darboux sum or the lower sum of f corresponding to the partion P, denoted by $L(P,f)$, is defined as $L(P,f) = \sum\limits_{r=1}^{n} m_r(x_r - x_{r-1})$

From (i), $m(x_r - x_{r-1}) \le m_r(x_r - x_{r-1}) \le M_r(x_r - x_{r-1}) \le M(x_r - x_{r-1})$

$\therefore m \sum\limits_{r=1}^{n}(x_r - x_{r-1}) \le \sum\limits_{r=1}^{n} m_r(x_r - x_{r-1}) \le \sum\limits_{r=1}^{n} M_r(x_r - x_{r-1}) \le M \sum\limits_{r=1}^{n}(x_r - x_{r-1})$

or, $m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$ ... (ii)

From (ii), we see that $\{U(P,f) : P \in \mathcal{P}[a,b]\}$ and $\{L(P,f) : P \in \mathcal{P}[a,b]\}$ are bouned sets

The lower integral of f on $[a,b]$, denoted by $\underline{\int_a^b} f dx$, is defined as

$$\underline{\int_a^b} f dx = \sup_{P \in \mathcal{P}[a,b]} L(P,f)$$

The upper integral of f on $[a,b]$, denoted by $\overline{\int_a^b} dx$, is defined as $\overline{\int_a^b} = \inf\limits_{P \in \mathcal{P}[a,b]} U(P,f)$

Books followed: 1. Real Analysis – S.K. Mapa
2. Advances Mathematical Analysis – Utpal Chatterjee
3. Principles of Mathematical Analysis – W. Rudin
4. Mathematical Analysis – S.C. Malik & S. Arora
5. Theories of Integration – D.S. Kurtz and C.W. Swartz

Note: from (i), $L(P,f) \le U(P,f)$ for a partition P of $[a,b]$ where $f:[a,b]\to\mathbb{R}$ is a bounded function on $[a,b]$

Lemma 1.3.1 Let $f:[a,b]\to\mathbb{R}$ be a bounded function on $[a,b]$ and P be a partition of $[a,b]$. If Q be a refinement of P then
$U(P,f) \ge U(Q,f)$ and $L(P,f) \le L(Q,f)$

Proof: Let $P=(x_0, x_1, \ldots, x_n)$. We first examine the effect of adjoining one additional point y to P.

Let $P_1 = (x_0, x_1, \ldots, x_{k-1}, y, x_k, \ldots, x_n)$

Let $M_r = \sup\limits_{x\in[x_{r-1},x_r]} f(x)$ $\quad m_r = \inf\limits_{x\in[x_{r-1},x_r]} f(x)$, $r=1,2,\ldots,n$

The subinterval $[x_{k-1}, x_k]$ is divided into two smaller subintervals $[x_{k-1}, y]$ and $[y, x_k]$ in P_1

Let $M_k' = \sup\limits_{x\in[x_{k-1},y]} f(x)$ $\quad m_k' = \inf\limits_{x\in[x_{k-1},y]} f(x)$

$M_k'' = \sup\limits_{x\in[y,x_k]} f(x)$ $\quad m_k'' = \inf\limits_{x\in[y,x_k]} f(x)$

Then $M_k' \le M_k$, $M_k'' \le M_k$; $m_k' \ge m_k$, $m_k'' \ge m_k$

$\therefore M_k(x_k - x_{k-1}) = M_k(x_k - y) + M_k(y - x_{k-1}) \ge M_k''(x_k - y) + M_k'(y - x_{k-1})$

and $m_k(x_k - x_{k-1}) = m_k(x_k - y) + m_k(y - x_{k-1}) \le m_k''(x_k - y) + m_k'(y - x_{k-1})$

$\therefore U(P,f) - U(P_1,f) = M_k(x_k - x_{k-1}) - M_k'(y - x_{k-1}) - M_k''(x_k - y) \ge 0$

$L(P,f) - L(P_1,f) = m_k(x_k - x_{k-1}) - m_k'(y - x_{k-1}) - m_k''(x_k - y) \le 0$

$\therefore U(P,f) \ge U(P_1,f)$ and $L(P,f) \ge L(P_1,f)$

If Q be a refinement of P, then Q can be obtained from P by adjoining a finite number of additional points to P, one at a time.

By repeating the argument a finite number of times, we have
$U(P,f) \ge U(Q,f)$ and $L(P,f) \le L(Q,f)$.

This completes the proof.

Note: As $L(P,f) \le U(P,f)$, it follows that,

$$L(P,f) \le L(Q,f) \le U(Q,f) \le U(P,f)$$

Also $U(Q,f) - L(Q,f) \le U(P,f) - L(P,f)$

1.4. Norm of a partition: Let $[a,b]$ be a closed and bounded and $P = (x_0, x_1, \dots, x_n)$ be a partition of $[a,b]$
The norm of the partition P, denoted by $\|P\|$, is defined as

$$\|P\| = \max \{ x_r - x_{r-1} : r = 1,2,3,\dots,n \}$$

Note: If Q be a refinement of P then $\|Q\| \le \|P\|$. But the converse is not true, let $P = (0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1)$ and $Q = (0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1)$
P and Q are partitions of $[0,1]$. Then $\|P\| = \frac{1}{4}$ and $\|Q\| = \frac{1}{6}$ $\therefore \|Q\| \le \|P\|$ But Q is not a refinement of P.

Lemma 1.4.1. Let $f : [a,b] \to \mathbb{R}$ be bounded on $[a,b]$ and P be a partition of $[a,b]$ with $\|P\| = \delta$. If P_k be a refinement of P with k additional points of partition, then

$$0 \le U(P,f) - U(P_k, f) \le (M-m)k\delta,$$

and $$0 \le L(P_k, f) - L(P,f) \le (M-m)k\delta,$$

where $M = \sup_{x \in [a,b]} f(x)$, $m = \inf_{x \in [a,b]} f(x)$

Proof: Let $P = (x_0, x_1, \dots, x_n)$. First, we examine the effect of adjoining one additional point y to P and let

$$P_1 = (x_0, x_1, \dots, x_{k-1}, y, x_k, \dots, x_n)$$

Let $M_r = \sup_{x \in [x_{r-1}, x_r]} f(x)$ and $m_r = \inf_{x \in [x_{r-1}, x_r]} f(x)$, for $r = 1,2,\dots,n$

The subinterval $[x_{r-1}, x_r]$ is divided into small subintervals $[x_{k-1}, y]$ and $[y, x_k]$

Let $M_k' = \sup_{x \in [x_{k-1}, y]} f(x)$ and $m_k' = \inf_{x \in [x_{k-1}, y]} f(x)$