Since the series $\left[1+\frac{1}{2}+\frac{1}{3}+\cdots\right]$ is a divergent series, it follows from (ii) that $\lim_{\epsilon\to 0}\int_\epsilon^1 f(x)dx$ is not finite and consequently the integral $\int_0^1 |f|(x)\,dx$ is divergent.

Note: This example establishes that the converse of Theorem 2.4.1 is not true.

B. Improper integrals on an unbounded interval

2.5 Definition: I. Convergence of the improper integral $\int_a^\infty f(x)dx$ where f is integrable on $[a, X]$ $\forall X > a$

Let $\phi(X) = \int_a^X f(x)\,dx$, $X > a$

If $\lim_{X\to\infty}\phi(X)$ exists (finitely) then the improper integral $\int_a^\infty f(x)dx$ is said to be convergent. If the limit be l, we write $\int_a^\infty f(x)\,dx = l$

If $\lim_{X\to\infty}\phi(X)$ does not exist (finitely) then the improper integral $\int_a^\infty f(x)dx$ is said to be divergent.

Examples: 1. Let us consider the integral $\int_0^\infty e^{-x}dx$. The integrand is integrable on any closed interval $[0, X]$, $X>0$. The integral is improper. Let $\phi(X) = \int_0^X e^{-x}dx$, $X>0$.

Then $\phi(X) = 1 - e^{-X}$. $\lim_{X\to\infty}\phi(X) = 1$. So, $\int_0^\infty e^{-x}dx$ is convergent and $\int_0^\infty e^{-x}dx = 1$

2. Let us consider the integral $\int_0^\infty \frac{1}{1+x}dx$. The integrand is integrable on any closed interval $[0, X]$, $X>0$

The integral is improper. Let $\phi(X) = \int_0^X \frac{1}{1+x}dx$, $X > 0$

Then $\phi(X) = \log(1+X)$ $\lim_{X\to\infty} \log(1+X) = \infty$. So,

$\int_0^\infty \frac{1}{1+x}dx$ is divergent.

II. Convergence of the improper integral $\int_{-\infty}^b f(x)dx$ where f is integrable on $[X, b]$ $\forall X < b$.

Let $\phi(X) = \int_X^b f(x)dx$, $X < b$. If $\lim_{X\to-\infty} \phi(X)$ exists (finitely) then the improper integral $\int_{-\infty}^b f(x)dx$ is said to be convergent. If the limit be l, we write

$$\int_{-\infty}^b f(x)dx = l$$

III Convergence of the improper integral $\int_{-\infty}^\infty f(x)dx$ where f is integrable on $[X_1, X_2]$ $\forall X_1, X_2 \in \mathbb{R}$ satisfying $X_1 < X_2$

Let $c \in \mathbb{R}$. If both $\int_{-\infty}^c f(x)dx$ and $\int_c^\infty f(x)dx$ be convergent according to the definitions I and II above, then improper integral $\int_{-\infty}^\infty f(x)$ is said to be convergent and we write $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$

Examples (contd.) 3. Let us consider the integral $\int_{-\infty}^\infty \frac{dx}{1+x^2}$.

The integrand is integrable on any closed interval $[x_1, x_2]$, $x_2 > x_1$. The integral is improper

Let us consider the integrals $\int_{-\infty}^a \frac{dx}{1+x^2}$ and $\int_a^\infty \frac{dx}{1+x^2}$ where at $\mathbb{R}$. Let $f(x) = \int_x^a \frac{dx}{1+x^2}$, $x < a$

Then $f(x) = \tan^{-1} a - \tan^{-1} x$. $\lim_{x \to -\infty} f(x) = \tan^{-1} a + \pi/2$

So, $\int_{-\infty}^{a} \frac{dx}{1+x^2}$ is convergent. Let $\psi(x) = \int_a^x \frac{dx}{1+x^2}$, $x > a$

Then $\psi(x) = \tan^{-1} x - \tan^{-1} a$ $\lim_{x \to \infty} \psi(x) = \pi/2 - \tan^{-1} a$

So, the improper integral $\int_a^\infty \frac{dx}{1+x^2}$ is convergent.

Consequently, the integral $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ is convergent and

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \left(\tan^{-1} a + \pi/2\right) + \left(\pi/2 - \tan^{-1} a\right) = \pi.$$

IV Convergence of the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ when f has a finite number of points of infinite discontinuities $c_1, c_2, \ldots, c_m$.

Let $c_1 < c_2 < \cdots < c_m$. If each of the integral $\int_{-\infty}^{c_1} f(x)\,dx$, $\int_{c_1}^{c_2} f(x)\,dx, \ldots \int_{c_{m-1}}^{c_m} f(x)\,dx$ and $\int_{c_m}^{\infty} f(x)\,dx$ be convergent according to the definition given earlier, then the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ is said to be convergent and we write

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \cdots + \int_{c_{m-1}}^{c_m} f(x)\,dx + \int_{c_m}^{\infty} f(x)\,dx$$

2.6 Tests for convergence, positive integrand

Theorem 2.6.1 Let a function f be integrable on $[a, X]$ $\forall X > a$ and $f(x) > 0$ $\forall x \geq a$.

A necessary and sufficient condition for the convergence of the improper integral $\int_a^\infty f(x)\,dx$ is that $\exists$ a positive real number k such that $\int_a^X f(x)\,dx < k \ \forall X > a$.

Theorem 2.6.2 Comparison test
Let the functions f and g be both integrable on $[a, X]$ $\forall X > a$ and $0 < f(x) \leq k\, g(x)$ $\forall x \geq a$, where $k > 0$. Then

(i) $\int_a^\infty g(x)\,dx$ is convergent $\Rightarrow \int_a^\infty f(x)\,dx$ is convergent

(ii) $\int_a^\infty f(x)\,dx$ is divergent $\Rightarrow \int_a^\infty g(x)\,dx$ is divergent.

Theorem 2.6.3 Comparison test (limit form)
Let the function f and g be both integrable on $[a, X]$ $\forall X > a$ and $f(x) > 0$, $g(x) > 0$ $\forall x \geq a$. If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = l$, where l is a non-zero finite number then the two improper integrals $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ converge or diverge together.

Note 1. If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, then for a preassigned positive ϵ $\exists$ a positive real number $b > a$ such that $f(x) < \epsilon\, g(x)$ $\forall x > b$
Then $\int_a^\infty g(x)\,dx$ is convergent implies $\int_a^\infty f(x)\,dx$ is convergent.

2. If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$ then for a preassigned positive G $\exists$ a positive real number $b > a$ such that $f(x) > G\, g(x)$ $\forall x > b > a$
Then $\int_a^\infty g(x)\,dx$ is divergent implies $\int_a^\infty f(x)\,dx$ is divergent.

2.6.4 A useful comparison integral
The improper integral $\int_a^\infty \frac{dx}{x^\mu}$, where $a > 0$, is convergent if and only if $\mu > 1$

2.6.5 μ test (A practical test) : Let $f(x) > 0$ $\forall x \geq a$.
If $\lim_{x\to\infty} x^\mu f(x) = l$, where l is a non-zero finite number, the improper integral $\int_a^\infty f(x)\,dx$ is convergent if and only if $\mu > 1$.