Worked Examples : 1. Examine the convergence of the improper integral

$$\int_1^\infty \frac{1}{x(1+x^2)}\,dx$$

Solution: Let the given integral be $\int_1^\infty f(x)\,dx$. Then $f(x) > 0$ for all $x \geq 1$. Let $g(x) = \frac{1}{x^3}$. Then $\int_1^\infty g(x)\,dx$ is convergent and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$, a non-zero finite number. By Comparison test, $\int_1^\infty f(x)\,dx$ is convergent.

2. Prove that the integral $\int_0^\infty x^{m-1} e^{-x}\,dx$ is convergent if and only if $m > 0$

Solution: Let the given integral be $\int_0^\infty f(x)\,dx$. If $m \geq 1$, 0 is not a point of infinite discontinuity of f. f has an infinite discontinuity if $m < 1$

Convergence at 0. $(m < 1)$

$f(x) > 0 \quad \forall x \in (0,1]$. Let $g(x) = x^{m-1}$, $x \in (0,1]$. Then $g(x) > 0 \quad \forall x \in (0,1]$ and $\lim_{x\to 0} \frac{f(x)}{g(x)} = 1$, a non-zero finite number.

$\int_0^1 g(x)\,dx$ is convergent if and only if $1-m < 1$, i.e., if and only if $m > 0$

By comparison test, $\int_0^1 f(x)\,dx$ is convergent if and only if $m > 0$

Convergence at ∞ : $f(x) > \quad \forall x \geq 1$. Let $g(x) = \frac{1}{x^2}$, $x \geq 1$.

Then $g(x) > 0 \quad \forall x \geq 1$ and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{m+1}}{e^x} = 0 \quad \forall m$

As the integral $\int_1^\infty g(x)\,dx$ is convergent, therefore the integral $\int_1^\infty x^{m-1} e^{-x}\,dx$ is convergent $\forall m$

Hence the given integral is convergent if and only if $m > 0$

Note. The integral $\int_0^\infty x^{m-1} e^{-x}\,dx$, $m > 0$ is called the Gamma function, and is denoted by $\Gamma(n)$

Definition: The improper integral $\int_a^\infty f(x)\,dx$ is said to be absolutely convergent if the integral $\int_a^\infty |f|(x)\,dx$ be convergent.

The improper integral $\int_{-\infty}^\infty f(x)\,dx$ is said to be absolutely convergent if the integral $\int_{-\infty}^\infty |f|(x)\,dx$ be convergent.

Theorem 2.6.6 An absolutely convergent improper integral $\int_a^\infty f(x)\,dx$ (where f is bounded and integrable on $[a, X]$, for every $X > a$) is convergent.

Worked Example (Continues)

3. Examine the convergence of the improper integral $\int_1^\infty f(x)\,dx$

where $f(x) = \frac{1}{x^2}$, if x be rational ≥ 1
$= -\frac{1}{x^2}$ if x be rational >1

Solution: $|f|(x) = \frac{1}{x^2}$, $x \geq 1$. $\int_1^\infty |f|(x)\,dx$ is convergent and therefore $\int_1^\infty f(x)\,dx$ is absolutely convergent. Consequently, the integral $\int_1^\infty f(x)\,dx$ is convergent.

Theorem 2.6.7 (Abel's test for convergence on the integral of a product)
Let (i) a function ϕ be monotonic and bounded on $[a, \infty)$
and (ii) the integral $\int_a^\infty f(x)$ be convergent
Then the integral $\int_a^\infty f(x)\phi(x)\,dx$ is convergent.

Theorem 2.6.8 (Dirichlet's test for convergence on the integral of a product)
Let (i) a function ϕ be monotonic and bounded on $[a, \infty)$
and $\lim_{x\to\infty} \phi(x) = 0$ and
(ii) the integral $\int_a^X f(x)\,dx$ is bounded on $[a, X]$ $\forall X > a$
Then the integral $\int_a^\infty f(x)\phi(x)\,dx$ is convergent.

Worked Example (continued)

4. Show that the improper integral $\int_0^\infty \frac{\sin x}{x} dx$ is convergent.

Proof: Since $\lim_{x \to 0+} \frac{\sin x}{x} = 1$, 0 is not a point of infinite discontinuity of the integrand. So, $\int_0^1 \frac{\sin x}{x} dx$ is convergent ---(i)

Let us consider the improper integral $\int_1^\infty \frac{\sin x}{x} dx$.

Let $f(x) = \sin x,\ x \geq 1$: $g(x) = \frac{1}{x},\ x \geq 1$. Then g is bounded and monotone decreasing function on $[1, \infty)$ and $\lim_{x \to \infty} g(x) = 0$. $\left| \int_1^X f(x)dx \right| = \left| -\cos X + \cos 1 \right| < 2$. So, $\int_1^X f(x)dx$ is bounded on $[1, X]\ \forall X > 1$.

By Dirichlet's test $\int_1^\infty \frac{\sin x}{x} dx$ is convergent ---(ii)

From (i) and (ii) it follows that $\int_0^\infty \frac{\sin x}{x} dx$ is convergent.

2.7. Beta and Gamma Function:

The improper integral $\int_0^1 x^{m-1}(1-x)^{n-1} dx$ is convergent if $m > 0$, $n > 0$. The integral $\int_0^1 x^{m-1}(1-x)^{n-1} dx$, $m > 0, n > 0$ is called the Beta function and is denoted by $B(m, n)$

Thus $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

The improper integral $\int_0^\infty e^{-x} x^{n-1} dx$ is convergent if $n > 0$. The integral $\int_0^\infty e^{-x} x^{n-1} dx$, $n > 0$ is called the Gamma function and it is denoted by $\Gamma(n)$

Thus $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx,\ n > 0$

Properties 1. $B(1,1) = 1$

Proof: $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx,\ m>0, n>0$

So, $B(1, 1) = \int_0^1 dx = 1$

2. $B(m,n) = B(n,m)$

Proof: $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$, $m>0, n>0$

$= \lim_{\epsilon\to 0, \delta\to 0} \int_\epsilon^{1-\delta} x^{m-1}(1-x)^{n-1}dx$

Let $x = 1-y$. Then $dx = -dy$

$\lim_{\epsilon\to 0, \delta\to 0} \int_\epsilon^{1-\delta} x^{m-1}(1-x)^{n-1}dx = \lim_{\epsilon\to 0, \delta\to 0} \int_\delta^{1-\epsilon} (1-y)^{m-1} y^{n-1} dy$

$= \lim_{\delta\to 0, \epsilon\to 0} \int_\delta^{1-\epsilon} y^{n-1}(1-y)^{m-1} = \int_0^1 y^{n-1}(1-y)^{m-1}dy = B(n,m)$

3. $B(m+1,n) = \frac{m}{m+n} B(m,n)$, $m>0, n>0$

Proof: $B(m+1,n) = \int_0^1 x^{m} (1-x)^{n-1} dx$

$= \left[\frac{x^m(1-x)^n}{-n}\right]_0^1 + \frac{m}{n}\int_0^1 x^{m-1}(1-x)^n dx$

$= \frac{m}{n}\int_0^1 (1-x)\, x^{m-1}(1-x)^{n-1}dx$

$= \frac{m}{n}\int_0^1 x^{m-1}(1-x)^{n-1}dx - \frac{m}{n}\int_0^1 x^m (1-x)^{n-1}dx$

$= \frac{m}{n} B(m,n) - \frac{m}{n} B(m+1,n)$

So, $\left(1+\frac{m}{n}\right) B(m+1,n) = \frac{m}{n} B(m,n)$

Or, $B(m+1,n) = \frac{m}{m+n} B(m,n)$

4. $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$, $m>0$, $n>0$

Proof: $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$, $m>0$, $n>0$

Let $x = \sin^2\theta$. Then $dx = 2\sin\theta\cos\theta\, d\theta$

As $x\to 0+$, $\theta\to 0+$ as $x\to 1-$, $\theta\to \frac{\pi}{2}-$

So, $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-2}\theta \cos^{2n-2}\theta \sin\theta\cos\theta\, d\theta$

$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta\, d\theta$, $m>0$, $n>0$