Deductions:

(i) $\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{1}{2} B\left(\frac{m+1}{2}, \frac{n+1}{2}\right), m>-1, n>-1$

(ii) $\int_0^{\pi/2} \sin^n\theta \, d\theta = \int_0^{\pi/2} \cos^n\theta \, d\theta = \frac{1}{2} B\left(\frac{n+1}{2}, \frac{1}{2}\right), n>-1$

(iii) $B\left(\frac{1}{2}, \frac{1}{2}\right) = 2\int_0^{\pi/2} d\theta = \pi$

5. $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx, \quad m>0, n>0$

Proof: $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx, \quad m>0, n>0$

Let $x = \frac{t}{1+t}$. Then $dx = \frac{1}{(1+t)^2} dt$

As $x \to 0+$, $t \to 0+$; as $x \to 1-$, $t \to \infty$

Therefore $B(m,n) = \int_0^\infty \left(\frac{t}{1+t}\right)^{m-1} \left(\frac{1}{1+t}\right)^{n-1} \frac{1}{(1+t)^2} dt$

$= \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}} dt = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

6. $B(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$

Proof: we have $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

Let $x = \frac{1}{t}$ in the second integral. Then $dx = -\frac{1}{t^2} dt$

As $x \to 1+$, $t \to 1-$; as $x \to \infty$ $t \to 0+$

$\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^1 \frac{1}{t^{m-1}} \cdot \frac{t^{m+n}}{(1+t)^{m+n}} \cdot \frac{1}{t^2} dt$

$$= \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}} dt = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

So, $$B(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$= \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$$

7. $\Gamma(1) = 1$

Proof: $\Gamma(1) = \int_0^\infty e^{-x} dx = \lim_{X\to\infty} \int_0^X e^{-x} dx = \lim_{X\to\infty} \left[1 - e^{-X}\right]_0^X = 1$

8. $\Gamma(n+1) = \Gamma(n)$, $n>0$

Proof: $$\int_\varepsilon^X x^n e^{-x} dx = \left[\frac{x^n e^{-x}}{-1}\right]_\varepsilon^X + n\int_\varepsilon^X x^{n-1} e^{-x} dx$$

$$= -X^n e^{-X} + \varepsilon^n e^{-\varepsilon} + n\int_\varepsilon^X x^{n-1} e^{-x} dx$$

Proceeding to limit as $X \to \infty$ and $\varepsilon \to 0$, we have

$$\int_0^\infty e^{-x} x^n dx = n\int_0^\infty e^{-x} x^{n-1} dx$$

or $\Gamma(n+1) = n\Gamma(n)$, $n>0$

Corollary: If $n$ be a positive integer then $\Gamma(n+1) = n!$

$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n(n-1)\ldots 2.1\,\Gamma(1) = n!$

~~Also~~ We assume without proof the result

9. $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, $m>0$, $n>0$

Deductions: (i) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$B\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(1)} = \left(\Gamma(\tfrac{1}{2})\right)^2$$

So, $\left(\Gamma(\tfrac{1}{2})\right)^2 = B(\tfrac{1}{2}, \tfrac{1}{2}) = \pi$ and this gives $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

(ii) If $m, n$ be positive integers, $B(m+1, n+1) = \dfrac{m!\, n!}{(m+n+1)!}$

$$B(m+1, n+1) = \frac{\Gamma(m+1)\,\Gamma(n+1)}{\Gamma(m+n+2)}, \quad m > -1,\ n > -1$$

If $m, n$ are positive integers $\Gamma(m+1) = m!$, $\Gamma(n+1) = n!$

and $\Gamma(m+n+2) = (m+n+1)!$

So, $$B(m+1, n+1) = \frac{m!\, n!}{(m+n+1)!}$$

10. Legendre's Duplication formula:

$$\sqrt{\pi}\,\Gamma(2n) = 2^{2n-1}\,\Gamma(n)\,\Gamma(n+\tfrac{1}{2}), \quad n > 0$$

Proof: $$\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = B(m,n)$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta, \quad m>0,\ n>0 \quad \cdots (i)$$

Taking $m = n$, we have, $$\frac{(\Gamma(n))^2}{\Gamma(2n)}$$

$$= 2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2n-1}\theta\, d\theta$$

$$= \frac{1}{2^{2n-2}} \int_0^{\pi/2} \sin^{2n-1} 2\theta\, d\theta$$

$$= \frac{1}{2^{2n-1}} \int_0^{\pi} \sin^{2n-1}\phi\, d\phi \quad (\text{Taking } 2\theta = \phi)$$

$$= \frac{1}{2^{2n-1}} \cdot 2 \int_0^{\pi/2} \sin^{2n-1}\phi\, d\phi \quad \cdots (ii)$$

Taking $m = \frac{1}{2}$ in (i), we have $$\frac{\Gamma(n)\,\Gamma(\tfrac{1}{2})}{\Gamma(n+\tfrac{1}{2})} = 2\int_0^{\pi/2} \cos^{2n-1}\theta\, d\theta$$

$= 2\int_0^{\pi/2} \sin^{2n-1}\theta\, d\theta \quad \ldots \text{(iii)}$

From (ii) & (iii), we have

$$\frac{(\Gamma(n))^2}{\Gamma(2n)} = \frac{1}{2^{2n-1}} \frac{\Gamma(n)\Gamma(\tfrac{1}{2})}{\cancel{\Gamma((2n+1)/2)}\,\Gamma(n+\tfrac{1}{2})}, \quad n>0$$

or, $\sqrt{\pi}\,\Gamma(2n) = 2^{2n-1}\Gamma(n)\,\Gamma(n+\tfrac{1}{2})$, since $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$.

We assume another result ~~which is~~ (without proof)

11. $\Gamma(m)\,\Gamma(1-m) = \dfrac{\pi}{\sin m\pi}, \quad 0<m<1$

12. (i) $\displaystyle\int_0^\infty e^{-kt} t^{n-1}\, dt = \frac{\Gamma(n)}{k^n}, \quad k>0,\ n>0$

(ii) $\displaystyle\int_1^\infty \frac{(\log y)^{n-1}}{y^{k+1}} = \frac{\Gamma(n)}{k^n}, \quad k>0,\ n>0$

Proof: (i) $\displaystyle\int_0^\infty e^{-kt} t^{n-1}\, dt$

$\displaystyle= \int_0^\infty e^{-y}\left(\frac{y}{k}\right)^{n-1}\cdot\frac{1}{k}\, dy$ [Let $y = kt$. As $t\to\infty$, $y\to\infty$ since $k>0$]

$\displaystyle= \frac{1}{k^n}\int_0^\infty e^{-y} y^{n-1}\, dy, \quad n>0$

$\displaystyle= \frac{\Gamma(n)}{k^n}$

(ii) $\displaystyle\int_1^\infty \frac{(\log y)^{n-1}}{y^{k+1}}\, dy$

$\displaystyle= \int_0^\infty t^{n-1} e^{-kt}\, dt$ [Let $\log y = t$. Then $y = e^t$, $y=1 \Rightarrow t=0$]

$\displaystyle= \frac{\Gamma(n)}{k^n}, \quad k>0,\ n>0$ [using (i)]