3. Sequence of functions

3.1 Definition: Let D be a subset of $\mathbb{R}$ and for each $n \in \mathbb{N}$, let $f_n : D \to \mathbb{R}$ be a function. Then $\{f_n\}$ is a sequence of functions on D to $\mathbb{R}$. D is said to be the domain of the sequence of functions $\{f_n\}$

3.2 Pointwise Convergence: Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}$, let $f_n : D \to \mathbb{R}$ be a function. The sequence $\{f_n\}$ is said to be pointwise convergent on D if for each $x \in D$, the sequence $\{f_n(x)\}$ converges.

Let the sequence $\{f_n\}$ be pointwise convergent on D and let $x \in D$. Then $\{f_n(x)\}$ is convergent. Let $\lim f_n(x) = l_x$. Since $\forall x \in D$, $\{f_n(x)\}$ converges to a limit, l_x exists $\forall x \in D$. Let us define a function $f : D \to \mathbb{R}$ by

$$f(x) = l_x, \quad x \in D.$$

Then f is said to be the limit function of the sequence $\{f_n\}$ on D. In this case we say the sequence $\{f_n\}$ converges to f on D and we write $f = \lim f_n$ on D or $f_n \to f$ on D.

Examples 1. For each $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = x^n$, $x \in \mathbb{R}$. Then $\{f_n\}$ is a sequence of functions on $\mathbb{R}$. For each $x \in (-1, 1)$ the sequence $\{f_n(x)\}$ converges to 0 and for $x = 1$, the sequence $\{f_n(x)\}$ converges to 1. For all other $x \in \mathbb{R}$, the sequence $\{f_n(x)\}$ is not convergent.

So, the sequence $\{f_n\}$ is pointwise convergent on $(-1, 1]$ and the limit function f is defined by

$$f(x) = 0, \quad -1 < x < 1$$
$$= 1, \quad x = 1$$

2. (i) For each $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{nx}{1+n^2x^2}$, $x \in \mathbb{R}$. Then $\{f_n\}$ is a sequence of function on $\mathbb{R}$.

For $x=0$, the sequence is $\{0, 0, \cdots 0\}$. This converges to 0

For $x \neq 0$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = 0$.

So, the sequence $\{f_n\}$ is pointwise convergent on $\mathbb{R}$, to the function f defined by $f(x) = 0$, $x \in \mathbb{R}$.

3. Let $f_n(x) = \tan^{-1} nx$, $x \in \mathbb{R}$

Then
$$\lim_{n\to\infty} f_n(x) = \frac{\pi}{2}, \text{ if } x>0$$
$$= 0, \text{ if } x=0$$
$$= -\pi/2, \text{ if } x<0$$

So, the sequence $\{f_n\}$ is pointwise convergent on $\mathbb{R}$ to the limit function f defined on $\mathbb{R}$ by
$$f(x) = \pi/2, x>0$$
$$= 0, x=0$$
$$= -\pi/2, x<0$$

3.3. Uniform convergence: Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}$, let $f_n : D \to \mathbb{R}$ be a function. Then sequence $\{f_n\}$ is said to be ~~unifor~~ uniformly convergent on D to a function f if corresponding to a preassigned $\varepsilon > 0$, $\exists$ a natural number ~~k~~ $k(\varepsilon)$ (depending on ε but not on $x \in D$) such that $|f_n(x) - f(x)| < \varepsilon \;\; \forall n \geq k$, $\forall x \in D$,

In this case we write $\lim f_n = f$ uniformly on D or $f_n \to f$ uniformly on D. f is said to be the uniform limit of the sequence $\{f_n\}$ on D.

Examples: 1. In example 1 of 3.2, the sequence $\{f_n\}$ converges on $(-1, 1]$ to the function $f = 0, \ -1 < x < 1$
$= 1, \ x = 1$

Let us examine if the convergence of the sequence $\{f_n\}$ is uniform on $(0, 1)$.

Let $c \in (0,1)$. Then $|f_n(c) - f(c)| = c^n$

Let $0 < \varepsilon < 1$ Then $|f_n(c) - f(c)| < \varepsilon$ whenever $c^n < \varepsilon$

i.e whenever $n \log(1/c) > \log(1/\varepsilon)$

i.e whenever $n > \log(1/\varepsilon) / \log(1/c)$

Let $k = [\log(1/\varepsilon)/\log(1/c)] + 1$ [$[x]$ = integral part of x]

Then k is a natural number and

$|f_n(c) - f(c)| < \varepsilon \ \forall n \geq k$

So, for all $x \in (0,1)$, $|f_n(x) - f(x)| < \varepsilon \ \forall n \geq k$,

where $k = [\log(1/\varepsilon)/\log(1/x)] + 1$. This k depends on ε as well as on x. As $x \to 1-$, $k \to \infty$

It follows that there does not exist a natural number k such that $\forall x \in (0,1)$, $|f_n(x) - f(x)| < \varepsilon$ holds $\forall n \geq k$. Consequently, $\{f_n\}$ is not uniformly convergent on $(0,1)$

Let $a \in \mathbb{R}$ such that $0 < a < 1$

In $[0, a]$, the greatest value of $\log(1/\varepsilon)/\log(1/x)$ is $\log(1/\varepsilon)/\log(1/a)$. Let $k = [\log(1/\varepsilon)/\log 1/a] + 1$

Then k is a natural number and $\forall x \in [0, a]$,

$|f_n(x) - f(x)| < \varepsilon \ \forall n \geq k$

This proves that the sequence $\{f_n\}$ is uniformly convergent on $[0, a]$

Example 2. In example 4 of 3.2, the sequence $\{f_n\}$ converges on $\mathbb{R}$ to the function f where $f(x)=0$, $x\in\mathbb{R}$
Let us examine if the convergence of the sequence is uniform on $[0,\infty)$. For all $x\geq 0$, $|f_n(x)-f(x)| = \frac{nx}{1+n^2x^2}$.
Let $u(x)=\frac{nx}{1+n^2x^2}$ for $x>0$. Then $u'(x) = \frac{n(1-n^2x^2)}{(1+n^2x^2)^2}$.

$u'(x)>0$ for $x<\frac{1}{n}$, $u'(x)=0$ for $x=1/n$,

$u'(x)<0$ for $x>\frac{1}{n}$

u is a maximum at $x=\frac{1}{n}$ and $u(\frac{1}{n})=\frac{1}{2}$

i.e., $|f_n(\frac{1}{n})-f(\frac{1}{n})|=\frac{1}{2}$

Let $\varepsilon=\frac{1}{4}$. If the sequence $\{f_n\}$ be uniformly convergent on $[0,\infty)$ to the function f, then for the chosen ε there must exist a natural number k such that $\forall x\geq 0$, $|f_n(x)-f(x)|<\frac{1}{4}$ holds $\forall$ $n\geq k$. But for every natural number k, $|f_k(1/k)-f(1/k)|=\frac{1}{2} \not< 1/4$
This shows that no natural number k can be found so that $\forall x\in[0,\infty)$, $|f_n(x)-f(x)|<\frac{1}{4}$ holds $\forall n\geq k$. So, $\{f_n\}$ is not uniformly convergent on $[0,\infty)$. Let $a\in\mathbb{R}$ such that $a>0$
For all $x>0$, $|f_n(x)-f(x)| = \frac{nx}{1+n^2x^2} < \frac{1}{nx}$. Then $\forall x\geq a$,

$|f_n(x)-f(x)| < \frac{1}{na}$

Let us choose $\varepsilon>0$. If $k=\left[\frac{1}{a\varepsilon}\right]+1$, then k is a natural number and $\forall x\geq a$, $|f_n(x)-f(x)|<\varepsilon$ $\forall n\geq k$.

This proves that the sequence is uniformly convergent on $[a,\infty)$.