It follows that if the sequence $\{f_n\}$ is uniformly convergent on D to the function f then the sequence $\{f_n\}$ also converges pointwise on D to f. But the converse is not true, is discussed in the following examples (see Page-91, Page-92 & Page-93)

[3.3 should be before Page-90]

Theorem 3.3.1 (Cauchy's Criterion) Let $D \subset \mathbb{R}$ and let $\{f_n\}$ be a sequence of functions on D to $\mathbb{R}$. A necessary and sufficient condition for uniform convergence of the sequence $\{f_n\}$ on D is that for a preassigned positive ε, $\exists$ a natural number k such that $\forall x \in D$,

$$\left| f_{n+p}(x) - f_n(x) \right| < \varepsilon \quad \forall\, n \geq k \text{ and } p = 1, 2, 3, \ldots$$

Proof: Let the sequence $\{f_n\}$ be uniformly convergent on D and let the limit function be f. Then for a preassigned positive ε $\exists$ a natural number k (depending on ε) such that $\forall x \in D$, $\left| f_n(x) - f(x) \right| < \varepsilon/2 \quad \forall\, n \geq k$

So, if $p = 1, 2, 3, \ldots$ then $\forall x \in D$

$$\left| f_{n+p}(x) - f(x) \right| < \varepsilon/2 \text{ holds } \forall\, n \geq k$$

Then $\forall x \in D$

$$\left| f_{n+p}(x) - f_n(x) \right| \leq \left| f_{n+p}(x) - f(x) \right| + \left| f_n(x) - f(x) \right| < \varepsilon \quad \forall\, n \geq k,\ p = 1, 2, 3, \ldots$$

Conversely, let the condition be satisfied. Then for a chosen $\varepsilon > 0$ $\exists$ a natural number k such that $\forall x \in D$, $\left| f_{n+p}(x) - f_n(x) \right| < \varepsilon \quad \forall\, n \geq k,\ p = 1, 2, 3, \ldots$

Let $x_0 \in D$. Then $\left| f_{n+p}(x_0) - f_n(x_0) \right| < \varepsilon \quad \forall\, n \geq k$ and $p = 1, 2, 3, \ldots$

It follows that the sequence $\{f_n(x_0)\}$ is a Cauchy sequence in $\mathbb{R}$ and hence it is convergent

Consequently, the sequence $\{f_n\}$ is pointwise convergent on D. Let the limit function be f

Let us choose $\varepsilon > 0$. Then by the condition, $\exists$ a natural number k (depending only on ε) such that $\forall x \in D$,

$$|f_{n+p}(x) - f_n(x)| < \varepsilon/2 \quad \forall n \geq k \text{ and } p = 1, 2, 3, \ldots$$

So, $\forall x \in D$, $f_k(x) - \varepsilon/2 < f_{k+p}(x) < f_k(x) + \varepsilon/2$ for $p = 1, 2, 3, \ldots$

Since $\lim_{p\to\infty} f_{k+p}(x) = f(x)$, taking limit as $p \to \infty$, we have $\forall x \in D$, $f_k(x) - \varepsilon/2 \leq f(x) \leq f_k(x) + \varepsilon/2$

or, $|f_k(x) - f(x)| \leq \varepsilon/2 < \varepsilon \ \forall x \in D$

Similar inequalities for $k+1, k+2, \ldots$

So, $\forall x \in D$, $|f_n(x) - f(x)| < \varepsilon \ \forall n \geq k$

This proves that $\{f_n\}$ is uniformly convergent to f on D.

Equivalent statement of for Cauchy Criterion

A necessary and sufficient condition for uniform convergence of a sequence $\{f_n\}$ on D is that for a preassigned $\varepsilon > 0$, $\exists$ a natural number k such that $\forall x \in D$, $|f_m(x) - f_n(x)| < \varepsilon \ \forall m, n \geq k$

Worked Examples

1. A sequence of functions $\{f_n\}$ is defined on $[0, a]$, $0 < a < 1$, by $f_n(x) = x^n$, $x \in [0, a]$. Show that $\{f_n\}$ converges uniformly on $[0, a]$

Solution: Let us choose $\varepsilon > 0$, such that $0 < \varepsilon < 2$

For all $x \in [0, a]$ and $\forall m, n \in \mathbb{N}$

$$|f_m(x) - f_n(x)| = |x^m - x^n| \leq |x|^m + |x|^n$$

$\le a^n + a^n \le 2a^m$ if $m \le n$

$|f_m(x) - f_n(x)| < \varepsilon$ holds if $a^m < \varepsilon/2$

i.e., if $m \log a < \log \varepsilon/2$

i.e, if $m > \dfrac{\log \varepsilon/2}{\log a}$, since $\log a < 0$

Let $K = \left[\dfrac{\log \varepsilon/2}{\log a}\right] + 1$. Then K is a natural number and $\forall x \in [0, a]$, $|f_m(x) - f_n(x)| < \varepsilon$ $\forall$ natural numbers m, n satisfying $n \geq m \geq K$.

By Cauchy's criterion, the sequence $\{f_n\}$ is uniformly convergent.

Theorem 3.3.2 Let $D \subset \mathbb{R}$ and let $\{f_n\}$ be a sequence of functions pointwise convergent on D to a function f. Let $M_n = \sup_{x \in D} |f_n(x) - f(x)|$. Then $\{f_n\}$ is uniformly convergent on D to f if and only if $\lim M_n = 0$

Proof: Let the sequence $\{f_n\}$ be uniformly convergent on D to f. Let $\varepsilon > 0$ be given. Then $\exists$ a natural number K (depending only on ε) such that $\forall x \in D$, $|f_n(x) - f(x)| < \varepsilon/2$ $\forall n \geq K$. This implies $\sup_{x \in D} |f_n(x) - f(x)| \le \frac{\varepsilon}{2} < \varepsilon$ $\forall n \geq K$

$M_n = |M_n| < \varepsilon$ $\forall n \geq K$. This proves that $\lim M_n = 0$

Conversely, let $\lim M_n = 0$. Let $\varepsilon > 0$ be given. Then $\exists$ a natural number k such that $|M_n| < \varepsilon$ $\forall n \geq k$.

or, $\sup_{x \in D} |f_n(x) - f(x)| < \varepsilon$ $\forall n \geq k$

So, $\forall x \in D$, $|f_n - f(x)| < \sup_{x \in D} |f_n(x) - f(x)| < \varepsilon$ $\forall n \geq k$

This proves that $\{f_n\}$ is uniformly convergent on D to f.

Worked Example (contd.)

2. A sequence of functions $\{f_n\}$ is defined by $f_n(x) = \frac{nx}{1+n^2x^2}$, $0 \le x \le 1$. Show that the sequence $\{f_n\}$ is not uniformly convergent on $[0,1]$

Solution: For $x=0$, the sequence is $\{0, 0, \ldots\}$. This converges to 0. For $0 < x \le 1$, $\lim_{n\to\infty} f_n(x) = 0$

Thus the sequence $\{f_n\}$ is convergent on $[0,1]$ and the limit function ~~is~~ f is defined by $f(x) = 0$, $0 \le x \le 1$

Let $M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)|$. Then $M_n = \sup_{x\in[0,1]} \frac{nx}{1+n^2x^2}$

For $x > 0$, $\frac{nx}{1+n^2x^2} \le \frac{1}{2}$ and $\frac{nx}{1+n^2x^2} = \frac{1}{2}$ at $x = \frac{1}{n}$

For $x = 0$; $\frac{nx}{1+n^2x^2} = 0$

Therefore, for $0 < x \le 1$, $\frac{nx}{1+n^2x^2} \le 1/2$

and $\frac{nx}{1+n^2x^2} = \frac{1}{2}$ at $x = \frac{1}{n}$

Clearly $\sup_{x\in[0,1]} \frac{nx}{1+n^2x^2} = \frac{1}{2}$. So, $M_n = \frac{1}{2}$ $\forall n \in \mathbb{N}$

Since $M_n = \frac{1}{2}$, so $\lim M_n = \frac{1}{2} \neq 0$.

So, the sequence $\{f_n\}$ is not uniformly convergent.

3. Let $f_n(x) = \frac{x}{n+x^2}$, $x \in [0,1]$. Show that $\{f_n\}$ is uniformly convergent on $[0,1]$.

Solution: The sequence converges to the function f where $f(x) = 0$, $x \in [0,1]$. Let $M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)|$

Then $M_n = \sup_{x\in[0,1]} \frac{x}{n+x^2}$