Let $u_n(x) = \frac{x}{n+x^2}$, $x \in [0,1]$ $u_n'(x) = \frac{n-x^2}{(n+x^2)^2} > 0$
$\forall x \in [0,1]$ and ~~[illegible]~~ and $\forall n > 1$

So, $\forall n > 1$, u_n is strictly increasing function of x on $[0,1]$ and so, $\sup_{x \in [0,1]} u_n(x) = \frac{1}{n+1}$

That is $M_n = \frac{1}{n+1}$ $\forall n > 1$, and $\lim M_n = 0$

Hence the sequence $\{f_n\}$ is uniformly convergent on $[0,1]$.

3.4 Some Consequences of uniform convergence

Theorem 3.4.1 Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}$, $f_n : D \to \mathbb{R}$ is bounded on D. If the sequence $\{f_n\}$ be uniformly convergent on D, then the limit function f is bounded on D.

Proof: Let us choose $\varepsilon > 0$. Since $\{f_n\}$ is uniformly convergent on D to f, $\exists$ a natural number k such that

$\forall x \in D$, $|f_n(x) - f(x)| < \varepsilon$ $\forall n \geqslant k$.

Let $\varepsilon = 1$. $\exists$ a natural number m such that $\forall x \in D$,
$|f_n(x) - f(x)| < 1$ $\forall n \geqslant m$. So, $\forall x \in D$,
$|f_m(x) - f(x)| < 1$.

Since $\big|\,|f(x)| - |f_m(x)|\,\big| \leq |f(x) - f_m(x)|$, it follows that $|f(x)| \leq |f_m(x)| + |f(x) - f_m(x)| < |f_m(x)| + 1$

Since f_m is bounded on D, $\exists$ a positive number B such that $|f_m(x)| < B$ $\forall x \in D$

So, $\forall x \in D$, $|f(x)| < B+1$ and this proves that f is bounded on D.

Theorem 3.4.2 Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}$, $f_n : D \to \mathbb{R}$ is continuous on D. If the sequence $\{f_n\}$ be uniformly convergent on D to a function f, then f is continuous on D.

Proof: Let $c \in D$. Let us choose $\varepsilon > 0$. Since $\{f_n\}$ is uniformly convergent on D to the function f, $\exists$ a natural number K such that $\forall x \in D$,

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3} \quad \forall n \geq K$$

So, $|f_K(x) - f(x)| < \frac{\varepsilon}{3} \quad \forall x \in D$ and

$|f_K(c) - f(c)| < \frac{\varepsilon}{3}$. Since f_K is continuous at c, $\exists$ a positive δ such that $|f_K(x) - f_K(c)| < \frac{\varepsilon}{3}$ $\forall x \in N(c, \delta) \cap D$ $\left(N(c,\delta) = (c-\delta, c+\delta) \right)$

By triangle inequality,

$$|f(x) - f(c)| \leq |f(x) - f_K(x)| + |f_K(x) - f_K(c)| + |f_K(c) - f(c)|$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \quad \forall x \in N(c,\delta) \cap D$$

That is, $|f(x) - f(c)| < \varepsilon \quad \forall x \in N(c,\delta) \cap D$

This proves that f is continuous at c. Since c is arbitrary, f is continuous on D.

Theorem 3.4.3 Let $I = [a, b]$ be a closed and bounded interval and for each $n \in \mathbb{N}$, $f_n : I \to \mathbb{R}$ be Riemann Integrable on I, If the sequence $\{f_n\}$ converges uniformly to a function f on I then f is Riemann Integrable on I and moreover, the sequence $\left\{ \int_a^b f_n \right\}$ converges to $\int_a^b f$

Proof: Let us choose $\varepsilon > 0$. Since $\{f_n\}$ is uniformly convergent on $[a, b]$ to the function f, $\exists$ a natural number k such that $\forall x \in [a, b]$, $|f_n(x) - f(x)| < \frac{\varepsilon}{4(b-a)}$ $\forall n \geq k$

So, $\forall x \in [a,b]$, $|f_k(x) - f(x)| < \frac{\varepsilon}{4(b-a)}$

or, $f_k(x) - \frac{\varepsilon}{4(b-a)} < f(x) < f_k(x) + \frac{\varepsilon}{4(b-a)}$ $\forall x \in [a,b]$ ... (i)

Since f_k is integrable on $[a,b]$, $\exists$ a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ such that $U(P, f_k) - L(P, f_k) < \frac{\varepsilon}{2}$ ...(ii)

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$ [As f_k is bounded, so f is bdd on $[a,b]$ from (i)]

$M_r' = \sup\limits_{x \in [x_{r-1}, x_r]} f_k(x)$, $m_r' = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$, $r = 1, 2, \ldots, n$

From (i) it follows that $m_r \geq m_r' - \frac{\varepsilon}{4(b-a)}$; $M_r \leq M_r' + \frac{\varepsilon}{4(b-a)}$

$$U(P,f) = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \cdots + M_n(x_n - x_{n-1})$$
$$\leq M_1'(x_1 - x_0) + M_2'(x_2 - x_1) + \cdots + M_n'(x_n - x_{n-1}) + \frac{\varepsilon}{4}$$

$$L(P,f) = m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(x_n - x_{n-1})$$
$$\geq m_1'(x_1 - x_0) + m_2'(x_2 - x_1) + \cdots + m_n'(x_n - x_{n-1}) - \frac{\varepsilon}{4}$$

So, $U(P,f) - L(P,f) \leq U(P, f_k) - L(P, f_k) + \frac{\varepsilon}{2}$

$< \varepsilon$ by using (ii)

This proves that f is Riemann integrable on $[a,b]$

<u>Second part</u> Let us choose $\varepsilon > 0$. Since the sequence $\{f_n\}$ converges uniformly to f on $[a,b]$, $\exists$ a natural number k such that $\forall x \in [a,b]$, $|f_n(x) - f(x)| < \frac{\varepsilon}{2(b-a)}$ $\forall n \geq k$

We have $\left|\int_a^b [f_n(x) - f(x)]dx\right| \leq \int_a^b |f_n(x) - f(x)| dx \leq \frac{\varepsilon}{2(b-a)} \cdot (b-a)$

i.e., $= \frac{\varepsilon}{2} < \varepsilon$ $\forall n \geq k$

or, $\left| \int_a^b f_n(x)dx - \int_a^b f(x)dx \right| < \varepsilon$ $\forall n \geq k$

That implies $\lim\limits_{n \to \infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$.

In other words, $\{\int_a^b f_n\}$ converges to $\int_a^b f$

This completes the proof.

Theorem 3.4.4 Let $\{f_n\}$ be a sequence of functions on $[a,b]$ such that for each $n \in \mathbb{N}$ $f_n'(x)$ exists $\forall x \in [a,b]$. If the sequence of derivatives $\{f_n'\}$ converges uniformly on $[a,b]$ to a function g and the sequence $\{f_n\}$ converges at least at one point $x_0 \in [a,b]$, then the sequence $\{f_n\}$ is uniformly convergent on $[a,b]$ and if the limit function be f then $f'(x) = g(x)$ $\forall x \in [a,b]$.

Proof: Let us choose $\varepsilon > 0$. Since the sequence $\{f_n'\}$ is uniformly convergent on $[a,b]$, $\exists$ a natural number K_1 such that $\forall x \in [a,b]$, $\left|f_{n+p}'(x) - f_n'(x)\right| < \frac{\varepsilon}{2(b-a)}$ $\forall n \geq K_1$

and $p = 1,2,3,\ldots$

Also since $\{f_n(x_0)\}$ is convergent, $\exists$ a natural number K_2 such that $\left|f_{n+p}(x_0) - f_n(x_0)\right| < \frac{\varepsilon}{2}$ $\forall n \geq K_2$ and $p = 1,2,3,\ldots$. Let $k = \max\{K_1, K_2\}$

Then $\forall x \in [a,b]$, $\left|f_{n+p}'(x) - f_n'(x)\right| < \frac{\varepsilon}{2(b-a)}$ $\forall n \geq k$ and $p = 1,2,3,\ldots$ and $\left|f_{n+p}(x_0) - f_n(x_0)\right| < \frac{\varepsilon}{2}$ $\forall n \geq k$ and $p = 1,2,3,\ldots$

Applying Lagrange's Mean Value Theorem to the function $f_{n+p} - f_n$ on $[x_0, x]$ or $[x, x_0]$ where $x \in [a,b]$,

$\left|f_{n+p}(x) - f_n(x) - f_{n+p}(x_0) + f_n(x_0)\right| = |x - x_0|\left|f_{n+p}'(\xi) - f_n'(\xi)\right|$

where $x_0 < \xi < x$ or $x < \xi < x_0$, as the case may be.

Now $\left|f_{n+p}'(\xi) - f_n'(\xi)\right| < \frac{\varepsilon}{2(b-a)}$ $\forall n \geq k$, and $p = 1,2,3,\ldots$

and $|x - x_0| < b - a$ $\forall x \in [a,b]$

It follows that $\forall x \in [a,b]$, $\left|f_{n+p}(x) - f_n(x) - f_{n+p}(x_0) + f_n(x_0)\right| < \frac{\varepsilon}{2}$