$\forall n \geq k$ and $p = 1, 2, 3, \ldots$

Using triangle inequality,

$\forall x \in [a,b]$, $|f_{n+p}(x) - f_n(x)| \leq |f_{n+p}(x) - f_n(x) - f_{n+p}(x_0) + f_n(x_0)|$

$+ |f_{n+p}(x_0) - f_n(x_0)| < \varepsilon \quad \forall n \geq k$ and $p = 1, 2, 3, \ldots$

This proves that the sequence $\{f_n\}$ is uniformly convergent on $[a,b]$.

<u>Second part</u> Let f be the uniform limit of the sequence $\{f_n\}$ on $[a,b]$. We now prove that $f'(x) = g(x)$, $x \in [a,b]$. Let $c \in [a,b]$

Let us define a sequence of functions $\{\phi_n\}$ on $D = [a,b] - \{c\}$

by $\phi_n(x) = \dfrac{f_n(x) - f_n(c)}{x - c}$, $x \in [a,b] - \{c\}$

Then for $x \in [a,b] - \{c\}$, $\lim_{n\to\infty} \phi_n(x) = \lim_{n\to\infty} \dfrac{f_n(x) - f_n(c)}{x - c}$

$$= \frac{f(x) - f(c)}{x - c}$$

Applying mean value theorem to the function $f_{n+p} - f_n$ on $[c, x]$ or $[x, c]$, we have

$$|f_{n+p}(x) - f_n(x) - f_{n+p}(c) + f_n(c)| = |x - c|\,|f_{n+p}'(\eta) - f_n'(\eta)|$$

where $c < \eta < x$ or $x < \eta < c$

Let us choose $\varepsilon > 0$. Since the sequence $\{f_n'\}$ converges uniformly to g on $[a,b]$, $\exists$ a natural number k such that

$\forall x \in [a,b]$, $|f_{n+p}'(x) - f_n'(x)| < \varepsilon \quad \forall n \geq k$, and $p = 1, 2, 3, \ldots$

So, $|f_{n+p}'(\eta) - f_n'(\eta)| < \varepsilon \quad \forall n \geq k$, and $p = 1, 2, 3, \ldots$

For all $x \in [a,b] - \{c\}$, $|\phi_{n+p}(x) - \phi_n(x)|$

$$= \left| \frac{f_{n+p}(x) - f_n(x) - f_{n+p}(c) + f_n(c)}{x - c} \right|$$

$$< \varepsilon \quad \forall n \geq k \text{ and } p = 1, 2, 3, \ldots$$

This proves that the sequence $\{f_n\}$ is uniformly convergent on D. Now c is a limit point of D.

Since the sequence $\{f_n\}$ is uniformly convergent on D, it follows that $\lim\limits_{n\to\infty}\lim\limits_{x\to c} f_n(x) = \lim\limits_{x\to c}\lim\limits_{n\to\infty} f_n(x)$ (we assume this result, we have not proved it here)

$$\lim_{n\to\infty}\lim_{x\to c} f_n(x) = \lim_{n\to\infty}\lim_{x\to c}\frac{f_n(x)-f_n(c)}{x-c} = \lim_{n\to\infty} f_n'(c) = g(c)$$

and $$\lim_{x\to c}\lim_{n\to\infty} f_n(x) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c} = f'(c)$$

So, $g(c) = f'(c)$. Since c is arbitrary

$f'(x) = g(x) \quad \forall x \in [a,b]$. This completes the proof.

Note 1. For a sequence of functions $\{f_n\}$ where each f_n is differentiable on $[a,b]$, mere uniform convergence of the sequence $\{f_n'\}$ on $[a,b]$ is not enough to ensure uniform convergence of the sequence $\{f_n\}$ on $[a,b]$

For example, let $f_n(x) = \log(n+x^2)$, $x\in[0,1]$

Then $f_n'(x) = \dfrac{2x}{n+x^2}$, $x\in[0,1]$. The sequence $\{f_n'\}$ converges to the function g where $g(x)=0$, $x\in[0,1]$

Let $M_n = \sup\limits_{x\in[0,1]}|f_n'(x) - g(x)|$. Then $M_n = \sup\limits_{x\in[0,1]}\dfrac{2x}{n+x^2}$

Let $u_n(x) = \dfrac{2x}{n+x^2}$, $x\in[0,1]$. Then $u_n'(x) = \dfrac{2n-2x^2}{(n+x^2)^2} > 0$ $\forall$ $x\in[0,1]$ and $\forall n>1$. So, $\forall n>1$, u_n is strictly increasing function of x on $[0,1]$ and

so, $\sup\limits_{x\in[0,1]} u_n(x) = \dfrac{2}{n+1}$ $\forall n>1$ and $\lim M_n = 0$

Hence $\{f_n'\}$ is uniformly convergent on $[0,1]$.

But the sequence $\{f_n\}$ is not even pointwise convergent on $[0,1]$.

Note 2 For a sequence of functions $\{f_n\}$ where each f_n is differentiable on $[a,b]$ and the sequence $\{f_n\}$ is pointwise convergent on $[a,b]$, the uniform convergence of the sequence $\{f_n'\}$ on $[a,b]$ is only a sufficient but not a necessary condition for the uniform convergence of $\{f_n\}$ on $[a,b]$.

For example, let $f_n(x) = x - \frac{x^n}{n}$, $x \in [0,1]$

$\lim_{n\to\infty} f_n(x) = x$, $x \in [0,1]$. Here the limit function is f where $f(x) = x$, $x \in [0,1]$

Let $M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)|$. Then $M_n = \frac{1}{n}$ and $\lim_{n\to\infty} M_n = 0$. This establishes uniform convergence of the sequence $\{f_n\}$ on $[0,1]$

$$f_n'(x) = 1 - x^{n-1}. \quad \lim_{n\to\infty} f_n'(x) = 1, \; 0 \le x < 1$$
$$= 0, \; x = 1$$

The limit function of the sequence $\{f_n'(x)\}$ is not continuous on $[0,1]$. As each $f_n'(x)$ is continuous on $[0,1]$ and the limit function is not continuous on $[0,1]$, the convergence of the sequence $\{f_n'(x)\}$ is not uniform on $[0,1]$

Thus the sequence $\{f_n\}$ is uniformly convergent on $[0,1]$ ~~in spite of~~ in spite of non-uniform convergence of the sequence $\{f_n'\}$ on $[0,1]$ and our assertion is established

Worked Example (contd.)

4. Let $f: \mathbb{R} \to \mathbb{R}$ be uniformly continuous. For each

natural number n, let $f_n(x) = f\left(x + \frac{1}{n}\right)$, $x \in \mathbb{R}$. Prove that the sequence $\{f_n\}$ is uniformly convergent on $\mathbb{R}$.

Solution: ~~For all~~ $\forall x \in \mathbb{R}$, $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} f\left(x+\frac{1}{n}\right)$
$= f(x)$, since f is continuous at x.

So, the sequence $\{f_n\}$ converges to function f on $\mathbb{R}$.
Let $\varepsilon > 0$. Since f is uniformly continuous on $\mathbb{R}$, $\exists$ a positive δ such that $\forall x, u \in \mathbb{R}$,

$|x-u| < \delta \Rightarrow |f(x) - f(u)| < \varepsilon$ ---(i)

$\exists$ a natural number k such that $0 < \frac{1}{n} < \delta$ $\forall n \geq k$
It follows from (i) that $\forall x \in \mathbb{R}$, $\left|f\left(x+\frac{1}{n}\right) - f(x)\right| < \varepsilon$ $\forall n \geq k$. That is, $\forall x \in \mathbb{R}$, $|f_n(x) - f(x)| < \varepsilon$ $\forall n \geq k$
This proves that the sequence $\{f_n\}$ is uniformly convergent on $\mathbb{R}$ to the function f.

5. Let $\{f_n\}$ be a sequence of functions on an interval I that converges uniformly on I to a continuous function f. Let $c \in I$ and $\{x_n\}$ is any sequence in I converging to c. Prove that $\lim\limits_{n\to\infty} f_n(x_n) = f(c)$

Solution: Let $\varepsilon > 0$. Since the sequence $\{f_n\}$ is uniformly convergent on I, $\exists$ a natural number k_1 such that
$\forall x \in I$, $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ $\forall n \geq k_1$
Since $x_n \in I$, $|f_n(x_n) - f(x_n)| < \frac{\varepsilon}{2}$ $\forall n \geq k_1$
Since f is continuous at c and $\lim x_n = c$, $\lim f(x_n) = f(c)$
So, $\exists$ a natural number k_2 such that $|f(x_n) - f(c)| < \frac{\varepsilon}{2}$ $\forall n \geq k_2$
Let $k = \max\{k_1, k_2\}$. Then $|f_n(x_n) - f(x_n)| < \frac{\varepsilon}{2}$ and $|f(x_n) - f(c)| < \varepsilon/2$ $\forall n \geq k$
by triangle inequality,
$|f_n(x_n) - f(c)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(c)| < \varepsilon$ $\forall n \geq k$
This implies $\lim\limits_{n\to\infty} f_n(x_n) = f(c)$.